SIX-FUNCTOR FORMALISMS

These are lecture notes from *Marc Hoyois* three lectures on *six-functor formalisms* during the 2023 Young Topologists Meeting. Redaction by Léo Navarro Chafloque.

The formalism of six operations germinated in Grothendieck's mind between 1956 and 1963, as he recalls in *Récoltes et semailles*.[1] This machinery subsumes behaviours of *functorial cohomology theories* : these six operations allow to capture different behaviours concerning cohomology theories, such as analogies of *Poincaré duality*, *base change formulas*, and more. In some geometric context and with a functorial type of cohomology in hand, achieving a six-functor formalism amounts to unleash the full power of cohomology that one knows from classical topological spaces.

These notes will be divided in three parts (which correspond to the three lectures given by Marc Hoyois). The first is a presentation of the six-functor formalism for topological spaces. The second one is of categorical nature and presents a functorial and neat definition of what is a six-functor formalism. In the last lecture, we introduce motivic spectra and explain how they provide an universal six-functor formalism for schemes.

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[1] *"La découverte progressive de ce formalisme de dualité et de son ubiquité s'est faite par une réflexion solitaire, obstinée et exigeante, qui s'est poursiuvie entre les années 1956 et 1963." [Gro22, pp. 415–416]*

Contents

1. Six-functor formalism in the context of topological spaces

In this section, we emphasize that we are taking as our geometric context topological spaces and continuous maps, *not* up to homotopy. The main reference for this chapter is [Vol23a].

1.1. Recollection on Poincaré Duality.

The formalism of six operations notably allows to formulate abstractly a "Poincaré duality" type theorem for "smooth objects" in a given context. Before going into more generality, we recall the most classical Poincaré duality theorem. We will also use this as an opportunity to review how the notions of "compact support cohomology" and Borel-Moore homology are defined. This will help us to built some intuition for the *exceptional functors* $f_!$ *and* $f^!$ that we will encounter soon.

Theorem 1.1 (Poincaré duality for compact oriented manidolds)**.** *Let X be a compact oriented topological manifold of dimension d. Let A be an abelian group. Then we have an isomorphism of graded abelian groups*

$$\mathrm{H}^{d-*}(X, A) \cong \mathrm{H}_*(X, A)$$

with the singular cohomology with coefficients in A on the left and the singular homology on the right.

This isomorphism comes from

$$C^*(X, A)[d] \cong C_*(X, A)$$

in $\mathcal{D}(\mathbb{Z})$, with singular cochains on the left and singular chains on the right.

Remark. Here, $\mathcal{D}(\mathbb{Z})$ denotes the stable ∞-category of $H\mathbb{Z}$-modules in the sense of [HA]. This ∞-category can be understood as the one presented by the model category of chain complexes of abelian groups with the projective model structure. The isomorphism in the theorem can be constructed as a quasi-isomorphism of chain complexes.

We will now explain the generalization of this theorem to arbitrary topological manifolds, so dropping "compact" and "oriented".

We will need some preliminary definitions and recolletions.

Definition 1.2 (Local system)**.** Let X be a locally simply-path connected topological space. A *local system* on X is a functor

$$\Pi_1(X) \to \mathrm{Ab}\,.$$

Remark. Here $\Pi_1(X)$ denotes the fundamental groupoid defined using pointed homotopy classes of paths and concatenation. For locally simply path-connected spaces this definition is equivalent to a locally constant sheaf of abelian groups. For more general spaces than locally simply path-connected spaces, the latter is the correct definition of local system.

Example-Definition (Orientation double cover)**.** Let X be a topological manifold. Note by $\widetilde{X}$ the two-sheeted orientation cover. (See for example [Hat02] for an explicit construction). It then corresponds to a functor by the Galois correspondence,

$$\Pi_1(X) \to \big\{\{a, b\}\big\}$$

where $\{\{a,b\}\}$ denotes the full subcategory of Set consisting of the a unique object which is a set with two elements that we denote a and b. Sending this set to the abelian group $\mathbb{Z}$ by $\{a,b\} \mapsto \langle a, b \mid ab\rangle$, we get a local system

$$\widetilde{\mathbb{Z}} \colon \Pi_1(X) \to \mathrm{Ab}\,.$$

If $A \colon \Pi_1(X) \to \mathrm{Ab}$ is any local system on X, we define $\widetilde{A}$ to be $A \otimes \widetilde{\mathbb{Z}}$.

Example 1.3. We have $\widetilde{\mathbb{Z}} \otimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$. This is because there is only one canonical generator of $\mathbb{Z}/2\mathbb{Z}$. As $(-1)(-1) = 1$, we have $\widetilde{\mathbb{Z}} \otimes \widetilde{\mathbb{Z}} \cong \mathbb{Z}$. A manifold is *orientable* if and only if $\widetilde{\mathbb{Z}}$ is the constant sheaf $\mathbb{Z}$.

We will now define appropriate analogues of chains and cochains with coefficients in a local system (this is needed for the general statment of Poincaré duality). We fix a topological manifold X and a local system $A \colon \Pi_1(X) \to \mathrm{Ab}$. These definitions (in particular with A being constant) are to be remembered as the most "usual case" of the (co)homologies that we will appear in a general six-functor formalism (namely *homology*, *cohomology*, *Borel–Moore homology* and *compact support cohomology*).

Definition 1.4 (Singular chains with coefficients in a local system)**.** Let $n \geq 0$. We define $C_n(X, A)$ to be the abelian group consisting of sums

$$\sum_{\sigma \colon \Delta^n \to X} a_\sigma \sigma$$

where $a_\sigma \in A(\sigma(\mathrm{bar}))$ and *only finitely many terms are non-zero.* Here, bar denotes the barycenter of Δ^n. The differential is defined analogously to the usual differential,

$$\partial(a_\sigma \sigma) = \sum_{i=0}^{n} (-1)^i A(\gamma_i)(a_\sigma) d_i^* \sigma.$$

Here, γ_i denotes the line from the barycenter of Δ^n to the barycenter of the i-th face. This gives a positively graded chain complex $C_*(X, A)$.

Definition 1.5 (Singular cochains with coefficients in a local system)**.** Let $n \geq 0$. We define $C^n(X, A)$ to be the abelian group of functions

$$\varphi \colon \mathrm{Top}(\Delta^n, X) \to \bigsqcup_{x \in X} A(x)$$

such that for a $\sigma \in \mathrm{Top}(\Delta^n, X)$, we have $\varphi(\sigma) \in A(\sigma(\mathrm{bar}))$. The co-differential is defined dually as the case above. We then a get a positively graded cochain complex (negatively graded chain complex) $C^*(X, A)$.

Remark. If we take A to be a trivial local coefficient system, meaning that the functor has constant value $A \in \mathrm{Ab}$ and that all paths are sent to the identity, we get usual singular chains and cochains complexes.

We now proceed to generalization which are pertinent in the non-compact case.

Definition 1.6 (Borel–Moore singular chains)**.** The chain complex $C_*^{BM}(X, A) \supset C_*(X, A)$ is defined using the same definition as above, except that *we do not require anymore that* *only finitely many a_σ are zero*, but an element $\sum a_\sigma \sigma \in C_n^{BM}(X, A)$ must satisfy that for every compact subset $K \subset X$ the following set

$$\{\sigma \mid a_\sigma \neq 0 \quad \sigma(\Delta^n) \cap K \neq \emptyset\}$$

is finite.

Definition 1.7 (Compactly supported cochains)**.** The compactly supported singular chain complex $C_c^*(X, A) \subset C^*(X, A)$ is defined as the sub-complex of φ's such that there exists a compact subspace $K \subset X$ with $\varphi_{X \setminus K} = 0$.

Remark. When X is compact, these correspond to the notions defined above.

With these definitions in hand, we can formulate the generalization of Poincaré duality.

Theorem 1.8 (Twisted Poincaré duality for arbitrary topological manifolds)**.** *Let X be a topological manifold. Let A a local system on X. We have isomorphisms in $\mathcal{D}(\mathbb{Z})$*

$$C_c^*(X, \widetilde{A})[d] \xrightarrow{\sim} C_*(X, A)$$

and

$$C^*(X, \widetilde{A})[d] \xrightarrow{\sim} C_*^{BM}(X, A)$$

1.2. **Grothendieck's key insight.** Grothendieck key idea in his yoga of *six operations* is that the fundamental object of a cohomology theory $\mathrm{H}^*_{\text{of some sort}}(-,-)$ is some kind of functorial association

$$\{\text{Geometric context}\} \to \mathrm{Cat}_\infty$$

of the form

$$X \mapsto \{\text{Natural coefficients for } \mathrm{H}^*_{\text{of some sort}}(X, -)\}.$$

The "functoriality" of this association should be one of the features that a six-functor formalism encodes.

1.2.1. *A list of example of coefficients.* We give a list of non-precise examples of "coefficients" for cohomology theories. We *do not claim* that every example listed here admits a six-functor formalism as will be defined later.

Example 1.9. Taking topological spaces as a geometric context, the natural coefficients for ordinary cohomology of abelian groups is

$$\mathrm{Sh}(X, \mathcal{D}(\mathbb{Z})).$$

So that the fundamental object should be the association

$$X \mapsto \mathrm{Sh}(X, \mathcal{D}(\mathbb{Z})).$$

For generalized cohomology theories, the association would then be

$$X \mapsto \mathrm{Sh}(X, \mathrm{Sp}).$$

Where Sp stands for the stable ∞-category of spectra. In the following subsections, we will investigate this example.

Remark. Here we use the meaning of sheaves with value in a ∞-category as defined in [HTT].

Example 1.10. If we take the category of schemes as a geometric context, and quasi-coherent cohomology which is a main tool in algebraic geometry, coefficients would be the stable ∞-category of quasi-coherent sheaves. Therefore the key association would be,

$$X \mapsto \mathrm{Qcoh}(X).$$

See [SchSix, Lecture VIII] for a development on this topic.

Example 1.11. In this example, we record some knowledge from Bhatt's lecture [PF-G].

- If we take as a geometric context qcqs schemes over a field of characteristic zero, and *de Rham cohomology*, the key association should be

  $$X \mapsto \mathrm{Crys}(X)$$

  where $\mathrm{Crys}(X)$ is the category of $\mathcal{D}$-modules. This category can also be realized as a category of quasi-coherent sheaves on a stack

  $$\mathrm{Qcoh}(X^{dr})$$

  namely the *de Rham stack* of X. See for example [PF-G, Chapter 2] or [LuDmod]. See [SchSix, Appendix to Lecture VIII] for a treatment of six operations for $\mathcal{D}$-modules.
- If we take as a geometric context bounded p-adic formal schemes and syntomic cohomology, the key association should be

  $$X \mapsto \text{F-Gauge}_{\mathbb{\Delta}}(X).$$

  This construction is developed at length in [PF-G], and is a more complicated analogue in mixed characteristic of the picture in characteristic zero and de Rham cohomology sketeched above. The ∞-category $\text{F-Gauge}_{\mathbb{\Delta}}(X)$ is also a category of quasi-coherent sheaves on a stack, which is denoted by X^{Syn}.

Remark. Realizing categories of coefficients as categories of quasi-coherent sheaves on some algebraico-geometric object place categories of quasi-coherent sheaves in a particular spot. Examples as above are called *geometrizations* in [PF-G]. This phenomenon is also described in [SchSix].[2]

Example 1.12. We record for example the following which is proposed as an exercise in [SchSix, Exercise 1.7]. If X is a locally closed subset of $\mathbb{R}^n$, and $\underline{X}$ denotes the functor on schemes taking a scheme S to the set of continuous maps $|S| \to X$, then $\underline{X}$ is a pro-étale algebraic space and we have

$$\mathrm{Sh}(X, \mathcal{D}(\mathbb{Z})) \cong \mathcal{D}_{qc}(\underline{X}).$$

[2]Citing Scholze: "A curious phenomenon is that in most cases the association $X \to D(X)$ can be factored as a composite $\mathcal{D}_{qc} \circ F$ where the first functor F takes any $X \in \mathcal{C}$ to some other kind of geometric object $F(X)$, sometimes a scheme but often rather a stack or even an analytic stack, and the second functor is the functor of taking the derived category of quasi-coherent sheaves on an analytic stack. This gives a more geometric perspective on a 6-functor formalisms, as a functor F between different kinds of geometric objects. " (End of lecture 1, [SchSix])

1.3. **Some point set topological definitions.** In this subsection, we (re)define some topological notions such as (quasi)-compactness, separatedness, in a categorical way. These definitions will be familiar for those who know algebraic geometry. One can compare the following definitions to the usual definitions in algebraic geometry, for example in [Stacks, Chapter 01H8 and Chapter 01QL].

Definition 1.13. Let X and S, S' be topological spaces and $f\colon X \to S$ is a continuous map. We say that,

- We say that f is *universally closed* if $\forall S' \to S$ the pullback map $f'\colon X' \to S'$ is closed.
- We say that f is *separated* if the diagonal map $\Delta\colon X \to X \times_S X$ is (universally) closed.
- We say that f is *proper* if f is universally closed and separated.
- We say that f is *locally proper* if $\forall x \in X$ and for all neighbourhoods $U \ni x$, there are neighbourhoods $A \subset U$ of X and B of $f(x)$ such that $f\colon A \to B$ is a proper map.
- We say that f is *smooth* if $\forall x \in X$ there is an open neighbourhood U of x and V of $f(x)$ such that $f(U) \subset V$ and

$$\begin{tikzcd} U \arrow[r, "\sim"] \arrow[dr, "f"'] & V \times D \arrow[d, "pr_1"] \\ & V \end{tikzcd}$$

  for D an open subspace of $\mathbb{R}^d$ for some $d \geq 0$.
- We say that f is *étale* if it is a local homeomorphism (smooth of relative dimension zero).

Remark. These definitions are to be thought as maps such that the fibers are of the corresponding type from the next definition, who links these notions to usual topological notions.

Definition 1.14. Let X be topological spaces and $f\colon X \to S$ is a continuous map. We say that,

- We say that X is *quasi-compact* if $X \to *$ is universally closed. This is equivalent to usual quasi-compactness by the *tube lemma*.
- We say that X is *Hausdorff* if the diagonal map $\Delta\colon X \to X \times X$ is separated. It follows quickly from definitions that this is equivalent to the usual notion.
- We say that f is *proper* or *compact* if $X \to *$ is proper. This is therefore equivalent to the usual *quasi-compact and Hausdorff* definition.
- We say that X is *locally proper* if $X \to *$ is locally proper. If X is in addition separated, this is equivalent to the usual notion of locally compactness.
- We say that X is a *manifold* if $X \to *$ is smooth.[3]

Remark. The following implications hold.

- Proper $\implies$ locally proper.

[3]This is more general than the usual notion of topological manifold, which usually requires second countability and separatedness. However every result exposed in these notes work in this generality.

- Smooth $\implies$ locally proper.
- Every map between locally compact Hausdorff spaces is locally proper and separated.

1.4. **Six-functor formalism for topological spaces.** In this section, we explain the appropriate functorialities of

$$X \mapsto \mathrm{Sh}(X, \mathrm{Sp}).$$

It will serve as an introduction to the six-functor formalism; we will present a formal development of the latter in the next section. The three following adjoint functors form the soul of a six-functor formalism.

(1) A map $f\colon X \to Y$ always gives

$$f^*\colon \mathrm{Sh}(Y, \mathrm{Sp}) \rightleftarrows \mathrm{Sh}(X, \mathrm{Sp})\colon f_*$$

with $f^* \dashv f_*$. We call this functors *pullback* and *pushforward*. The pushforward is defined by $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$. The pullback is the unique left adjoint.

(2) If f is locally proper, we have in addition

$$f_!\colon \mathrm{Sh}(X, \mathrm{Sp}) \rightleftarrows \mathrm{Sh}(Y, \mathrm{Sp})\colon f^!$$

with $f_! \dashv f^!$. These are called *exceptional direct image and exceptional inverse image.* If f is separated, we can define

$$f_!\mathcal{F}(V) = \varinjlim_{\substack{K \subset f^{-1}(V) \\ K \to V \\ \text{proper}}} \mathcal{F}_K(f^{-1}(V)),$$

where if $Z \subset X$ closed, we define $\mathcal{F}_Z(U)$ as the homotopy fiber of

$$\mathcal{F}(U) \to \mathcal{F}(U \cap (X \setminus Z)).$$

If f is not separated, we can proceed by descent using locally separatedness.

(3) $\mathrm{Sh}(X, \mathrm{Sp})$ is a symmetric monoidal closed ∞-category.

Remark. The six functors are

$$f^* \dashv f_* \quad f_! \dashv f^! \quad \otimes \dashv \mathrm{Hom}\,.$$

The first adjunction is basic and holds true if we replace Sp by a 1-presentable category for example. But for $f_!$ passing to the ∞-context is essential. Take the compact case where $f_! = f_*$. Then f_* will not be cocontinuous if $\mathcal{C} = \mathrm{Ab}$ – this functor is famously not right-exact. From the point of view of homological algebra, this is where all the cohomology comes from, *i.e.* the default of right-exactness of f_*. This is corrected when passing to derived categories.

1.4.1. *Six-functor compatibilities.* These functors also have the following compatibilities, which are a key part of the six operations. We list them now.

(1) *Compatibilty between $*$ and $!$ Base change formula.*

For every $f\colon X \to Y$ locally proper and every cartesian square

$$\begin{tikzcd} X' \arrow[r, "g_X"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g_Y"'] & Y \end{tikzcd}$$

we have a natural isomorphism $g_Y^* f_! \simeq f'_! g_X^*$. By adjunction, this is the same as $f^! g_{Y,*} \simeq g_{X,*} f'^!$.

(2) *Compatibilty between $*$ and $\otimes$.*
The functor f^* is symmetric monoidal.

(3) *Compatibilty between $!$ and $\otimes$. Projection formula.*
The functor $f_!$ is $\mathrm{Sh}(Y, \mathrm{Sp})$-linear, meaning that the *projection formula holds*

$$f_!(A \otimes f^* B) \simeq f_!(A) \otimes B.$$

Outside of these first key properties we also observe the subsequent properties.

(a) If f is locally proper and separated, there is a natural map

$$f_! \to f_*$$

which is an isomorphism in the case where f is proper.

(b) If $j\colon U \to X$ is étale (for example an open embedding), $j_!$ is the extension by zero functor, and therefore left adjoint to j^*.

In summary, we have the following duality between proper and étale maps.

- For a proper map
$$f^* \dashv f_* = f_! \dashv f^!$$
- For an étale map
$$f_! \dashv f^! = f^* \dashv f_*$$
- For a finite étale map (finite covering) we have $f_* = f_!$ and $f^* = f^!$ with both functors being right *and* left adjoint to each other.

For more on how *proper* and *étale* are cohomollogically dual, see [SchSix, Lecture VI]

1.4.2. *How to express (co)homologies?* We will now study the case of $f\colon X \to *$. We want to now make a link between this formalism and the definitions in the first subsection. For a locally contractible space X and for a local system $A\colon \Pi_1(X) \to A$ the *sheaf cohomology*

$$f_* A = \Gamma(X, A) \in \mathcal{D}(\mathbb{Z})_{\leq 0}$$

coincides with the usual singular cochain cohomology defined above

$$\Gamma(X, A) = C^*(X, A).$$

Moreover, definitions below will correspond to the one given above in terms of chains for a locally contractible space X.

For a symmetric monoidal ∞-category $\mathcal{C}$ we denote by $\mathrm{Pic}(\mathcal{C})$ the full subcategory of invertible objects with respect to the monoidal structure. For example if $\mathcal{C} = \mathrm{Sh}(X, \mathcal{D}(\mathbb{Z}))$, the sheaf $\mathbb{Z}[d]$ for any $d \in \mathbb{Z}$ or the local system $\widetilde{\mathbb{Z}}$ associated to the orientation double cover are examples invertible objects. Tensoring by such objects are interpreted as shifts and twists.

We now define the according generalization of cohomologies presented in the first subesection for an arbitrary $E \in \mathrm{Sp}$ and $\xi \in \mathrm{Pic}(\mathrm{Sh}(X, \mathrm{Sp}))$.

Definition 1.15 ((Co)homologies in a six-functor formalism)**.** Let X be a topological space and $f\colon X \to *$ denote the unique map to the point. We define

The **cohomology** of X with coefficients in E twisted by ξ

$$\mathrm{H}^*(X, E, \xi) = \Gamma(X, E, \xi) = f_*(\xi \otimes f^* E).$$

If X is locally proper, we further define the following.

The **homology** of X with coefficients in E twisted by ξ

$$\mathrm{H}_*(X, E, \xi) = f_!(\xi^{-1} \otimes f^! E).$$

The **cohomology with compact support** of X with coefficients in E twisted by ξ

$$\mathrm{H}^*_c(X, E, \xi) = \Gamma_c(X, E, \xi) = f_!(\xi \otimes f^* E).$$

The **Borel–Moore homology** of X with coefficients in E twisted by ξ

$$\mathrm{H}^{BM}_*(X, E, \xi) = f_*(\xi^{-1} \otimes f^! E).$$

Remark. Let A be a local system on X. The previous definition does not capture homology and cohomology with coefficients in a local system. We see a local system as locally constant sheaf of abelian groups and therefore as an object in the heart of $\mathrm{Sh}(X, \mathcal{D}(\mathbb{Z}))$. Here is how to define them.

$$\begin{aligned}
&\mathrm{H}^*(X, A) \coloneqq f_*(A \otimes f^*\mathbb{Z}) = f_* A && \text{cohomology}\\
&\mathrm{H}_*(X, A) \coloneqq f_!(A \otimes f^!\mathbb{Z}) && \text{homology}\\
&\mathrm{H}^*_c(X, A) \coloneqq f_!(A \otimes f^*\mathbb{Z}) = f_! A && \text{compact support cohomology}\\
&\mathrm{H}^{BM}_*(X, A) \coloneqq f_*(A \otimes f^!\mathbb{Z}) && \text{Borel-Moore homology}
\end{aligned}$$

In the end of this lecture, we explain how known theorems in cohomology of topological spaces are interpreted in this formalism.

1.4.3. *Dualizing sheaf and Serre duality.*

Definition 1.16 (Dualizing sheaf)**.** Let $f\colon X \to S$ be a locally proper map of topological spaces. The sheaf

$$\omega_f = f^!(\mathbb{1})$$

is called the dualizing sheaf of f. In the case of $f\colon X \to *$, we write ω_X.

In particular, for a compact topological space X, we get for a sheaf $\mathcal{F} \in \mathrm{Sh}(X\,\mathrm{Sp})$.

$$f_* \operatorname{Map}(\mathcal{F}, \omega_X) \cong \operatorname{Map}(f_*\mathcal{F}, \mathbb{S})$$

which is a topological version of *Serre-duality.*

Remark. If X is a manifold of dimension d, then

$$\omega_X \otimes \mathbb{Z} \cong \widetilde{\mathbb{Z}}[d],$$

where $\widetilde{\mathbb{Z}}$ is the orientation double cover local system defined above.

1.4.4. *Recollement.* Let X be a topological space. Denote by $U \subset X$ an open subset. Denote by Z its closed complement. Then we have the following *recollement* or *exact sequence of stable ∞-categories* (see [HA, A.8])

$$\begin{tikzcd} \mathrm{Sh}(Z,\mathrm{Sp}) \arrow[r, "\iota_!=\iota_*" description] & \mathrm{Sh}(X,\mathrm{Sp}) \arrow[l, bend right, "\iota^*"'] \arrow[l, bend left, "\iota^!"] \arrow[r, "j^!=j^*" description] & \mathrm{Sh}(U,\mathrm{Sp}) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

The additional data in the recollement is detailled in [HA, A.8]. This notably means that the central category can be reconstructed from the two edge ones.

1.4.5. *Künneth formula.* Let X be locally proper and Y be an arbitrary topological space. Then the *Künneth formula* categorifies to

$$- \boxtimes - = p_X^*(-) \otimes p_Y^*(-) \colon \mathrm{Sh}(X,\mathrm{Sp}) \otimes \mathrm{Sh}(Y,\mathrm{Sp}) \to \mathrm{Sh}(X \times Y,\mathrm{Sp}).$$

being an isomorphism.

1.4.6. *Verdier-Lurie duality.* See [HA, p. 5.5.5] for a detailed account. Let X be a locally compact Hausdorff space. Lurie's version of Verdier duality is a categorified version which can be expressed as the fact that

$$\mathrm{Sh}(X,\mathrm{Sp}) \otimes \mathrm{Sh}(X,\mathrm{Sp}) \xrightarrow{\otimes} \mathrm{Sh}(X,\mathrm{Sp}) \xrightarrow{f_!} \mathrm{Sp}$$

is a perfect pairing, and then gives an equivalence $\mathbb{D} \colon \mathrm{Sh}(X,\mathrm{Sp}) \to \mathrm{Co}\,\mathrm{Sh}(X,\mathrm{Sp})$.

1.4.7. *Relative generalized Poincaré duality.* Let $f \colon X \to S$ be smooth. Then the relative generalized version of Poincaré duality can be expressed as that the natural map[4]

$$\omega_f \otimes f^* \to f^!$$

is an isomoprhism, together with the fact that ω_f is invertible. In words, and in relation with definitions in the case of the point, this isomorphism relates Borel–Moore homology and a twisted and shifted version of the cohomology. Note that theorem 1.8 follows from this isomorphism using the interpretation of these (co)homologies in this six-functor formalism.

Proof sketch. By definition, a smooth map is locally on the source of the form

$$X \times \mathbb{R}^n \to X.$$

Using descent and Künneth formula, one then reduces to prove Poincaré duality in the case of

$$f \colon \mathbb{R} \to *.$$

A key fact here is the homotopy invariance of cohomology interpted as

$$p^* \colon \mathrm{Sh}(X,\mathrm{Sp}) \to \mathrm{Sh}(X \times \mathbb{R},\mathrm{Sp})$$

being fully-faithful. See [HA, Appendix A.2]. In particular we only need [HA, Lemma A.2.2] here.

[4] Using adjunction and the projection formula $f_!(f^!\mathbb{1} \otimes -) = f_!f^!\mathbb{1} \otimes -$, it comes from the co-unit $f_!f^!\mathbb{1} \otimes - \to -$.

Let $E \in \mathrm{Sp}$. First note that

$$\begin{aligned} f^! E(\mathbb{R}) &= \mathrm{Map}(\mathbb{1}, f^! E) \\ &= \mathrm{Map}(f_! \mathbb{1}, E) \\ &= \mathrm{Map}(\Gamma_c(\mathbb{R}, \mathbb{S}), E). \end{aligned}$$

We now compute $\Gamma_c(\mathbb{R}, \mathbb{S}) = f_! \mathbb{1}$.

$$\begin{aligned} \Gamma_c(\mathbb{R}, \mathbb{S}) &= \varinjlim_{[-r,r] \subset \mathbb{R}} \mathrm{fib}(\Gamma(\mathbb{R}, \mathbb{S}) \to \Gamma(\mathbb{R} \setminus [-r, r], \mathbb{S})) \\ &= \varinjlim_{[-r,r] \subset \mathbb{R}} \mathrm{fib}(\mathbb{S} \xrightarrow{\Delta} \mathbb{S} \oplus \mathbb{S}) \\ &= \mathrm{fib}(\mathbb{S} \xrightarrow{\Delta} \mathbb{S} \oplus \mathbb{S}) \\ &= \Omega \mathbb{S}. \end{aligned}$$

We claim that $\omega_{\mathbb{R}} = \mathbb{S}^1 = \Sigma \mathbb{S}$. We show that this is the case on global sections.

$$\begin{aligned} f^! \mathbb{1}(\mathbb{R}) &= \mathrm{Map}(\mathbb{1}, f^! \mathbb{1}) \\ &= \mathrm{Map}(f_! \mathbb{1}, \mathbb{S}) \\ &= \mathrm{Map}(\Omega \mathbb{S}, \mathbb{S}) \\ &= \Sigma \mathbb{S}. \end{aligned}$$

Now we want to compare $f^! E(\mathbb{R})$ to $(f^* E \otimes \omega_{\mathbb{R}})(\mathbb{R}) = (f^* \Sigma E)(\mathbb{R})$. We have

$$\begin{aligned} f^* \Sigma E(\mathbb{R}) &= \mathrm{Map}(\mathbb{1}, f^* \Sigma E) \\ &= \Sigma E \text{ (by homotopy invariance)} \\ &= \mathrm{Map}(\Omega \mathbb{S}, E). \end{aligned}$$

Therefore $f^! E$ and $f^* E \otimes \omega_R$ agree on global sections.

1.4.8. *Atiyah duality.* Let X and Y be C^1 manifolds. Let $f : X \to Y$ be a C^1-submersion. Then we can refine Poincaré duality specifying that

$$\omega_f \cong S^{T_f},$$

where S^{T_f} denotes the fiberwise one point compactification of the realtive tangent space $T_f \to X$. Therefore, we get a sphere-bundle on X. Looking locally at the associated pointed anima and then the associated spectra defines a sheaf of spectra on X, that we denoted by S^{T_f}.

1.4.9. *Lefschetz-Hopf trace formula.* Let X be a compact C^1-manifold, $f \colon X \to X$ a C^1-map with isolated fixed points. Then

$$\sum_i (-1)^i \operatorname{Tr}(f_{\mathrm{H}_i(X, \mathbb{Q})}) = \sum_{x \in \mathrm{Fix}(f)} \mathrm{sgn}(\det(\mathrm{id} - T_x f)).$$

This formula was one of Grothendieck's motivation to develop a six-functor formalism for schemes. Namely for $f = F$ being the Frobenius endomorphism this formula along with Poincaré duality would imply the first two of Weil conjectures.

2. Categorical aspects

In this lecture we will expose how to express in a compact way all functorialities and fundamental properties of six-functor formalisms that we encountered in the context of topological spaces, leading to a formal definition of the latter.

References for this lecture are [Man22, Appendix A.5], Scholze exposition of Mann's approach in [SchSix] and the work of Gaitsgory and Rozenblyum in [GR17, Part III], in particular for the $(\infty, 2)$-point of view. What we call here *categories of spans* are also called *categories of correspondences* in the above references.

2.1. **Categories of spans.** Let $\mathcal{C}$ be an ∞-category with finite products. In the context of the upcoming definition of six-functor formalisms, this category is to be thought as a geometric context. Denote by L and R (for *left* and *right*) classes of morphisms of $\mathcal{C}$ closed under composition, base change and contain all isomorphisms.

Informally $\mathrm{Span}_{L,R}(\mathcal{C})$ is the ∞-category whose objects are objects of $\mathcal{C}$ and morphisms between X and Y are given by spans

$$\begin{tikzcd} & Z \arrow[dl, "L"'] \arrow[dr, "R"] & \\ X & & Y \end{tikzcd}$$

where the right arrow is in R and the left one is in L. The composition of two arrows is given as the big triangle in the diagram below.

$$\begin{tikzcd} & & Z \times_Y W \arrow[dl] \arrow[dr] & & \\ & Z \arrow[dl] \arrow[dr] & & W \arrow[dl] \arrow[dr] & \\ X & & Y & & T \end{tikzcd}$$

Note that we use the stability by base change and composition of L and R here. Note also that even in the case where $\mathcal{C}$ is an ordinary 1-category, $\mathrm{Span}_{L,R}(\mathcal{C})$ is a (2,1)-category because pullbacks are only defined up to natural isomorphisms.

This is maybe enough for intuition, and also for the case where $\mathcal{C}$ is an ordinary 1-category, but we will explain the how to define precisely the ∞-category $\mathrm{Span}_{L,R}(\mathcal{C})$, because we want to treat the case where $\mathcal{C}$ is an ∞-category. Seeing

$$\mathrm{Cat}_\infty \subset \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Ani})$$

as *complete Segal anima* will be helpful.

In what follows we denote by Tw the *twisted arrow construction*, see for example [Ker, Tag 00AZ]. The only thing that we will use is that $\mathrm{Tw}(\Delta^n)$ is the category of pairs $\{(i,j) \in [n] \times [n] \mid i \leq j\}$ with arrows between $(i,j) \to (k,l)$ if and only if $i \leq k$ and

$j \geq k$. For $n = 2$ we get the poset

$$\begin{tikzcd}[column sep=small] & & (0,2) \arrow[dl] \arrow[dr] & & \\ & (0,1) \arrow[dl] \arrow[dr] & & (1,2) \arrow[dl] \arrow[dr] & \\ (0,0) & & (1,1) & & (2,2) \end{tikzcd}$$

The picture for a general $[n]$ is the obvious generalization of such a triangle of arrows. Accordingly to this picture, we will call an arrow $(i,j) \to (i,j')$ a *left arrow* and an arrow of the form $(i,j) \to (i',j)$ a *right arrow*. We also therefore call a *minimal square* a square of the form

$$\begin{tikzcd} & (i,j) \arrow[dl] \arrow[dr] & \\ (i,j-1) \arrow[dr] & & (i+1,j) \arrow[dl] \\ & (i+1,j-1) & \end{tikzcd}$$

Definition 2.1 (∞-categories of spans)**.** Let $\mathcal{C}$ be a category with finite products and L and R as above. The $(\infty,1)$-category $\mathrm{Span}_{L,R}(\mathcal{C})$ is the complete Segal anima defined by

$$[n] \mapsto \mathrm{Fun}_{L,R,\mathrm{cart}}(\mathrm{Tw}(\Delta^n), \mathcal{C}) \subset \mathrm{Fun}(\mathrm{Tw}(\Delta^n), \mathcal{C})$$

where $\mathrm{Fun}_{L,R,\mathrm{cart}}(\mathrm{Tw}(\Delta^n), \mathcal{C})$ consists of functors

- which send left arrows to arrows of L and right arrows to arrows of R and
- send minimal squares to pullbacks.

Remark. We make some basic remarks about the category of spans.

The ∞-category $\mathrm{Span}_{L,R}(\mathcal{C})$ has a symmetric monoidal structure defined by $X \otimes Y = X \times Y$.

We have canonical functors $\mathcal{C} \to \mathrm{Span}_{L,All}(\mathcal{C})$ defined by

$$\begin{tikzcd} & & X \arrow[dl, equal] \arrow[dr] & \\ (X \to Y) \arrow[r, mapsto] & X & & Y \end{tikzcd}$$

and $\mathcal{C}^{\mathrm{op}} \to \mathrm{Span}_{All,R}(\mathcal{C})$

$$\begin{tikzcd} & & X \arrow[dl] \arrow[dr, equal] & \\ (X \to Y) \arrow[r, mapsto] & Y & & X. \end{tikzcd}$$

Also, any morphism can be canonically decomposed as composition of a morphism in the image of $\mathcal{C}^{\mathrm{op}}$ and then by a morphism in the image of $\mathcal{C}$, as follows.

$$\begin{tikzcd}
 & & Z \arrow[dl, equal] \arrow[dr, equal] & & \\
 & Z \arrow[dl, "f"'] \arrow[dr, equal] & & Z \arrow[dl, equal] \arrow[dr, "g"] & \\
X & & Z & & Y
\end{tikzcd}$$

Therefore, we already begin to see how the category of spans helps to encode a functorial data which contains both covariant and contravariant information.

In what follows we will denote by $\mathrm{Span}_R(\mathcal{C}) = \mathrm{Span}_{All,R}(\mathcal{C})$.

Remark. The following flow of remarks will seem trivial, but they will allows us to see how the soon to be defined very concise notion of six-functor formalism captures the tensor product and the projection formula.

Note that in $(\mathcal{C}^{\mathrm{op}}, \times)$ any object is an E_∞-algebra with the diagonal. Note also that $\mathcal{C}^{\mathrm{op}} \to \mathrm{Span}_{All,R}(\mathcal{C})$ is symmetric monoidal by definition. Therefore any $X \in \mathrm{Span}_R(\mathcal{C})$ is an E_∞-algebra in $\mathrm{Span}_R(\mathcal{C})$ with respect to "the law"

$$\begin{tikzcd}
 & X \arrow[dl, "\Delta"'] \arrow[dr, equal] & \\
X \times X & & X.
\end{tikzcd}$$

Moreover, if $f : X \to Y$ is also in R, the fact that the following

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f\times \mathrm{id}"'] & Y \arrow[d, "\Delta"'] \\
Y \times X \arrow[r, "\mathrm{id}\times f"] & Y \times Y
\end{tikzcd}$$

is always a pullback square implies that the morphism

$$f^R = \begin{tikzcd}
 & X \arrow[dl, equal] \arrow[dr, "f"] & \\
X & & Y
\end{tikzcd}$$

is a morphism of Y-module in $\mathrm{Span}_R(\mathcal{C})$.

Remark. Note that if the following diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\Delta"'] & Y \arrow[d, "\Delta"'] \\
X \times X \arrow[r, "f\times f"] & Y \times Y
\end{tikzcd}$$

is a pullback (this is the case if and only if f is a monomorphism), then f^R is a morphism of *non-unital* algebras. (In the upcoming context of a six-functor formalism it means that under this hypothesis $f_!(A \otimes B) \cong f_!A \otimes f_!B$.)

2.2. **Formal definition of six-functor formalisms.** With this language setup and these few remarks we are now able to get the neat definition we were announcing.

Definition 2.2 (3-functor formalism)**.** A *3-functor formalism* on $(\mathcal{C}, R)$ as above is a right-lax symmetric monoidal functor

$$\mathcal{D}\colon \operatorname{Span}_R(\mathcal{C}) \to \operatorname{Cat}_\infty .$$

We let the reader appreciate how concise the definition is. Note that stating this in the context of ∞-categories encodes all the wanted higher coherences. Moreover, let us unwind how we relate this definition to the six-functor formalism that we saw in the context of topological spaces.

- Of course, for an object $X \in \mathcal{C}$, the ∞-category $\mathcal{D}(X)$ is to be interpreted as the category of *coefficients* of the incarnation of cohomology that we want to encode.
- *Pullback.* For a morphism $f\colon X \to Y$ in $\mathcal{C}$, we define $f^*\colon \mathcal{D}(Y) \to \mathcal{D}(X)$ as the image of

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, equal] & \\ Y & & X \end{tikzcd}$$

- *Exceptional direct image.* For a morphism $f\colon X \to Y$ in R, we define $f_!\colon \mathcal{D}(Y) \to \mathcal{D}(X)$ as the image of

$$\begin{tikzcd} & X \arrow[dl, equal] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

  Therefore the class R is to be interpreted as the analogue of locally proper maps in the topological context, or at least the class of maps where $f_!$ is defined.
- The functor $\mathcal{D}$ therefore sends a general morphism in $\operatorname{Span}_R(\mathcal{C})$

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

  to $g_! f^*\colon \mathcal{D}(X) \to \mathcal{D}(Y)$.
- *Base change formula.* Let $f\colon Z \to X$ a morphism and $g\colon Z \to Y$ a morphism in R. The following composition of arrows in $\operatorname{Span}_R(\mathcal{C})$

$$\begin{tikzcd} & & Z \times_Y X \arrow[dl, "f'"'] \arrow[dr, "g'"] & & \\ & Z \arrow[dl, equal] \arrow[dr, "g"] & & X \arrow[dl, "f"'] \arrow[dr, equal] & \\ Z & & Y & & X \end{tikzcd}$$

  and the functoriality implies the base change formula $f^* g_! \cong g'_! f'^*$.

- *Tensor product.* Note that we have not yet used the right-lax symmetric monoidal assumption on $\mathcal{D}$. This comes now. Namely, as we have noticed above, any object in $\mathrm{Span}_R(\mathcal{C})$ is canonically an E_∞-algebra with the diagonal. Therefore the right-lax symmetric structure implies that any $\mathcal{D}(X)$ is a symmetric monoidal category. Explicitly,
$$-\otimes -\colon \mathcal{D}(X)\times\mathcal{D}(X)\xrightarrow{\mathrm{lax}_{X,X}}\mathcal{D}(X\times X)\xrightarrow{\Delta^*}\mathcal{D}(X).$$
The unit is an object $\mathbb{1} : * \to \mathcal{D}(*)$ and for every object $p : X \to *$ the unit in $\mathcal{D}(X)$ is $p^*\mathbb{1}$. Note that we can express a posteriori the lax-structure as $(A, B) \mapsto A \boxtimes B := p_X^* A \otimes p_Y^* B$. This follows from the naturality of the lax-structure with the following diagram.
$$\begin{tikzcd}
\mathcal{D}(X)\times\mathcal{D}(Y) \arrow[r, "\mathrm{lax}_{X,Y}"] \arrow[d, "{(p_X^*,p_Y^*)}"'] & \mathcal{D}(X\times Y) \arrow[d, "(p_X\times p_Y)^*"] \arrow[dr, bend left, equal] & \\
\mathcal{D}(X\times Y)\times\mathcal{D}(X\times Y) \arrow[r, "\mathrm{lax}_{X\times Y,X\times Y}"] \arrow[rr, bend right=10, "-\otimes-"'] & \mathcal{D}(X\times Y\times X\times Y) \arrow[r, "\Delta^*"] & \mathcal{D}(X\times Y)
\end{tikzcd}$$
- *Pullback is symmetric monoidal.* It is also consequence of the right-lax symmetric monoidal structure because
$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Span}_R(\mathcal{C}) \xrightarrow{\mathcal{D}} \mathrm{Cat}_\infty$$
is a composition of a symmetric monoidal functor and a right-lax symmetric one.
- *Projection formula.* Let $f : X \to Y$ be a morphism in R. Note that $\mathcal{D}(X)$ is a $\mathcal{D}(Y)$-module using f^*. Moreover, we also explained in the remark preceding the definition that
$$\begin{tikzcd}
 & X \arrow[dl, equal] \arrow[dr, "f"] & \\
X & & Y
\end{tikzcd}$$
is a morphism of Y-module in $\mathrm{Span}_R(\mathcal{C})$. Therefore, as before the compatibility with the monoidal structure imposed implies that $f_!$ is $\mathcal{D}(Y)$-linear.

We can now define a six-functor formalism, as a condition on a 3-functor formalism.

Definition 2.3 (6-functor formalism)**.** A *6-functor formalism* on $(\mathcal{C}, R)$ is a 3-functor formalism
$$\mathcal{D}\colon \mathrm{Span}_R(\mathcal{C}) \to \mathrm{Cat}_\infty$$
such that

(1) for each object $X \in \mathcal{C}$ the symmetric monoidal category $\mathcal{D}(X)$ is closed,
(2) for each map $f : X \to Y$, the functor $f^* : \mathcal{D}(Y) \to \mathcal{D}(X)$ is a left adjoint and
(3) for each map $f : X \to Y$, the functor $f_! : \mathcal{D}(X) \to \mathcal{D}(Y)$ is a left adjoint.

Remark. We denote by Pr^L_∞ the ∞-category of presentable ∞-categories. In Pr^L_∞ we consider only functor that are *left adjoints*. See [HTT, section 5.5] for a detailed account on the subject. We recall that a commutative monoid in Pr^L_∞ with respect to the Lurie tensor product is a presentable symmetric monoidal closed ∞-category. We denote by Pr^L_∞ the associated ∞-category. We simply remark that a *6-functor formalism* on $(\mathcal{C}, R)$ with value in presentable ∞-categories is then simply a right-lax symmetric monoidal functor

$$\mathrm{Span}_R(\mathcal{C}) \to \mathrm{Pr}^L_\infty,$$

making in this case the definition even more concise.

2.2.1. *Künneth formula.* We say that $\mathcal{D}$ satisfies *Künneth formula* for $X, Y \in \mathcal{C}$ if the lax-structure $\mathrm{lax}_{X,Y}$ is an isomorphism. Note that in the case where $\mathcal{D}$ takes value in Pr_∞ as in the previous remark, the lax monoidal structure is the Lurie tensor product and the Künneth formula happens in this setting.

2.2.2. *Verdier duality.* In this remark, we explain how to express *Verdier duality* in the formalism and how it interchanges ! and $*$. For an object X, such that $p_X\colon X \to *$ is in R, the Verdier dual functor is defined to be $\mathbb{D}_X\colon \mathcal{D}(X) \to \mathcal{D}(X)^{\mathrm{op}}$

$$A \mapsto \mathrm{Hom}(A, p_X^! \mathbb{1})$$

where Hom is here used to designate the mapping object from the cartesian closed structure. We now show that for $f\colon X \to Y$ in R and $Y \to *$ in R, we have a natural isomorphism

$$\mathbb{D}_Y f_! \cong (f_*)^{\mathrm{op}} \mathbb{D}_X.$$

Proof. Let $A \in \mathcal{D}(X)$ and $B \in \mathcal{D}(Y)$. Using adjucntions and projection formula we get

$$\mathrm{Map}_{\mathcal{D}(Y)}(B, f_* \mathrm{Hom}(A, p_X^! \mathbb{1})) \cong \mathrm{Map}_{\mathcal{D}(*)}(p_{X,!}(f^* B \otimes A), \mathbb{1}) \cong \mathrm{Map}_{\mathcal{D}(*)}(p_{Y,!}(B \otimes f_! A), \mathbb{1}).$$

But also

$$\mathrm{Map}_{\mathcal{D}(Y)}(B, \mathrm{Hom}(f_! A, p_Y^! \mathbb{1})) \cong \mathrm{Map}_{\mathcal{D}(Y)}(B \otimes f_! A, p_Y^! \mathbb{1}) \cong \mathrm{Map}_{\mathcal{D}(*)}(p_{Y,!}(B \otimes f_! A), \mathbb{1}).$$

□

We say that Verdier duality holds for $A \in \mathcal{D}(X)$ if the natural map

$$A \to \mathbb{D}_X^2(A)$$

is an equivalence. For $X = *$, objects where Verdier duality holds are precisely dualizable objects. For sheaves on stratified spaces, Verdier duality holds for *constructible sheaves.* See [Vol23b].

2.2.3. *Poincaré duality.* Let $f\colon X \to Y$ be a map in R. Then note that from the co-unit $f_! f^! \to \mathrm{id}$, we get using the projection formula a natural transformation

$$f_!(f^! \mathbb{1} \otimes f^* -) = f_! f^!(\mathbb{1}) \otimes - \to -.$$

By adjunction we get a *Poincaré comparaison* natural map

$$f^! \mathbb{1} \otimes f^* \to f^!.$$

We write $\omega_f = f^! \mathbb{1}$. Following [SchSix, Lecture V] we say that Poincaré duality holds for f if

(1) the above *Poincaré comparaison* natural map is an isomorphism,
(2) the object $f^! \mathbb{1}$ is invertible,

(3) for every map $g : Y' \to Y$, if f' denotes the base change of the map the f then property (1) and (2) also holds for f' and the natural map
$$g'^{*}\omega_f \to \omega_{f'}$$
is an isomorphism.

With respect to a six-functor formalism $\mathcal{D}$ on $\mathcal{C}$, Scholze [SchSix, Lecture V] calls such maps *$\mathcal{D}$-cohomologically smooth.*

2.2.4. *Associated (co)homologies to a six-functor formalism.* We briefly explain how to retrieve (co)homology from a six-functor formalism
$$\mathcal{D} : \mathrm{Span}_R(\mathcal{C}) \to \mathrm{Cat}_\infty$$
as defined above.

(1) The Y-relative cohomology of $X \xrightarrow{f} Y$ with coefficients in $A \in \mathcal{D}(Y)$ is
$$\mathrm{H}^*_Y(X, A) \coloneqq f_*f^*A.$$

(2) If $f : X \to Y$ is in R, the Y-relative homology of $X \xrightarrow{f} Y$ with coefficients in $A \in \mathcal{D}(Y)$ is
$$\mathrm{H}_{*,Y}(X, A) \coloneqq f_!f^!A.$$

(3) If $f : X \to Y$ is in R, the Y-relative compact support cohomology of $X \xrightarrow{f} Y$ with coefficients in $A \in \mathcal{D}(Y)$ is
$$\mathrm{H}^*_{c,Y}(X, A) \coloneqq f_!f^*A.$$

(4) If $f : X \to Y$ is in R, the Y-relative Borel–Moore homology of $X \xrightarrow{f} Y$ with coefficients in $A \in \mathcal{D}(Y)$ is
$$\mathrm{H}^{BM}_{*,Y}(X, A) \coloneqq f_*f^!A.$$

Furthermore, we can encode the following functorialities. Let
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & X' \arrow[d, "h"] \\ & Y \end{tikzcd}$$
be a commutative diagram.

(5) Fixing $A \in \mathcal{D}(Y)$, we get a functor
$$\mathrm{H}^*_Y(-, A) \colon \mathcal{C}^{\mathrm{op}}_{/Y} \to \mathcal{D}(Y).$$
For $f : X \to X'$ in $\mathcal{C}_{/Y}$ the unit $1 \to f_*f^*$ induces the *contravariant functoriality of cohomology*
$$h_*h^* \to g_*g^* = h_*f_*f^*h^*.$$

(6) Fixing $A \in \mathcal{D}(Y)$, we get a functor
$$\mathrm{H}_{*,Y}(-, A) \colon \mathcal{C}_{/Y} \to \mathcal{D}(Y).$$
For $f : X \to X'$ in $\mathcal{C}_{/Y}$ the counit $f_!f^! \to 1$ induces the *covariant functoriality of homology*
$$g_!g^! = h_!f_!f^!h^! \to h_!h^!.$$

2.3. **Examples of six-functor formalisms.** In what follows we record a few examples of six-functor formalisms.

2.3.1. *Topological spaces and generalized cohomology theories.* If $\mathcal{A}$ is a stable bicomplete category (for example $\mathcal{A} = \mathrm{Sp}$) then with $\mathcal{C}$ being locally compact Hausdorff spaces and $R =$All,

$$X \mapsto \mathrm{Sh}(X, \mathcal{A})$$

is a six-functor formalism. See [Vol23a] for a detailled account. Taking $\mathcal{A} = \mathrm{Sp}$ we get that this formalism of six operations is the formalism associated to generalized cohomology theory of topological spaces.

2.3.2. *Non-abelian cohomology of anima.* For $\mathcal{C} = \mathrm{Ani}$ and $R =$All, the following

$$X \mapsto \mathrm{Ani}_{/X} \cong \mathrm{Fun}(X, \mathrm{Ani})$$

is a six-functor formalism. Namely for $f\colon X \to Y$

$$\begin{tikzcd} \mathrm{Ani}_{/X} \arrow[r, shift left=3] \arrow[r, shift right=3] & \mathrm{Ani}_{/Y} \arrow[l, "-\times_Y X"'] \end{tikzcd}$$

so that $f^* = f^!$ is the pullback (this is consistent with the fact that every morphism is étale in Ani). The base-change of formula is easily verified. The left adjoint to the pullback is simply the forgetful functor. Note that the fact that $- \times_Y X$ is also a left adjoint follows from the fact that colimits are universal and the adjoint functor theorem.

In the case that $Y = *$ this right adjoint is simply $\mathrm{Map}_{\mathrm{Ani}_{/X}}(X, -)$. In this 6-functor formalism, for $A \in \mathrm{Ani}$ the *homology of X with coefficients in A* is

$$X \times A$$

and the *cohomology of X with coefficients in A* is

$$\mathrm{Map}(X, A).$$

2.3.3. *Parameterized spectra (Genralized cohomology for anima).* Again for $\mathcal{C} = \mathrm{Ani}$ and $R =$All but this time with *parameterized spectra* as coefficients.

$$X \mapsto \mathrm{Fun}(X, \mathrm{Sp}).$$

This is the is the six-functor formalism of generalized cohomology theories for anima. Again $f_* = f_!$ for every morphism f. The case of the point can be phrased as

$$\begin{tikzcd} \mathrm{Fun}(X, \mathrm{Sp}) \arrow[r, shift left=4, "\varinjlim"] \arrow[r, shift right=4, "\varprojlim"'] & \mathrm{Sp} \arrow[l, "\mathrm{cst}"'] \end{tikzcd}$$

Therefore in this six functor formalism, for $E \in \mathrm{Sp}$ the *homology of X with coefficients in E* is

$$\Sigma^\infty_+ X \otimes E$$

and the *cohomology of X with coefficients in E* is

$$\mathrm{Map}(\Sigma^\infty_+ X, E).$$

2.3.4. *Étale cohomology with (pro-)finite coefficients.* For $\mathcal{C} = \mathrm{Sch}$ and R =morphisms locally of finite type, the following

$$X \mapsto \mathrm{Sh}_{\text{ét}}(X, \mathcal{D}(\mathbb{Z})_{\mathrm{tor}})$$

extend to a six-functor formalism. The category $\mathcal{D}(\mathbb{Z})_{\mathrm{tor}}$ is the full subcategory of $\mathcal{D}(\mathbb{Z})$ which is the kernel of rationalization. This is the six-functor formalism for étale cohomology with finite coefficients. See for example [SchSix, Appendix to Lecture VII]. This theory was first developed in [SGA4]. Taking an inverse limit of cohomology as captured in this formalism lead to the first definition of the so called *l-adic cohomology.* If $f\colon X \to \mathrm{Spec}(k)$ for k an algebraically closed field,

$$\mathrm{H}^*_{l-\mathrm{adic}}(X, \mathbb{Q}_l) \cong (\varprojlim_n f_* f^* \mathbb{Z}/l^n\mathbb{Z})[\frac{1}{l}].$$

Remark. In this remark, we outline a possible non-abelian cohomology of ∞-topoi and it's basic properties. We have not verified the validity of the base change formula in this setup. Let $\mathcal{C} = \mathcal{T}\mathrm{op}$ be the ∞-category of ∞-topoi. What follows is a generalization of the example with relative anima.

We say that a geometric morphism $f : \mathcal{X} \to \mathcal{Y}$ is an *essential geometric morphism* if f^* admits a left adjoint $f_!$. We want to take as coefficients $\mathcal{T}\mathrm{op} \to \mathrm{Cat}_\infty$ the infinity topos itself

$$\mathcal{X} \to \mathcal{X}$$

with R being essential geometric morphisms, and $f^* = f^!$ for every such morphism. Therefore essential geometric morphisms may play the role of étale morphisms. In the case of $\pi : \mathcal{X} \to *$ being essential, we call

$$\pi_! \mathbb{1} :\ = \Pi_\infty \mathcal{X} \in \mathrm{Ani}$$

the *shape* of X or the *étale fundamental* ∞*-groupoid of* X[5]. We Let $A \in \mathrm{Ani}$. The *homology of* $\mathcal{X}$ *with coefficients in* A would be

$$\pi_! \pi^* A = \pi_! \pi^* \varinjlim_A * \cong \varinjlim_A \Pi_\infty \mathcal{X} = A \times \Pi_\infty \mathcal{X}.$$

Note that from a topos theoritic point of view $\mathrm{Fun}(A, \mathrm{Ani}) \cong \mathrm{Ani}_{/A}$ is the *classyfing topoi* βA of A[6]. We can express what would be the *cohomology of* $\mathcal{X}$ *with coefficients in* A as

$$\pi_* \pi^* A = \mathrm{Map}_{\mathcal{X}}(\mathbb{1}, \pi^* A) = \mathrm{Map}(\Pi_\infty \mathcal{X}, A) = \mathrm{Fun}_{\mathcal{T}\mathrm{op}}(\mathcal{X}, \beta A)$$

For more on this subject and how shape theory generalizes Galois theory see [Hoy17].

One would also be more likely to try to approach such a development with stable coefficients

$$\mathrm{Sp}(\mathcal{X}) = \mathcal{X} \otimes \mathrm{Sp}$$

or more generally $\mathcal{X} \otimes \mathcal{A}$ for any stable and bi-complete ∞-category $\mathcal{A}$ in the spirit of [Vol23a]. In this context, we note that the notion of *proper morphism of* ∞*-topoi* [HTT, Section 7.3.1] could be pertinent in the development of such a 6-functor formalism, as

[5]This last name is motivated because this is a direct generalization of Grothendieck definition of étale fundamental group. It is constructed to be the homotopy type classifying torsors with constant coefficients.

[6]In the case where A is BG for G a discrete group, this is the ∞-toposic analogue of the classyfing topos of G.

base change formula will hold basically by definition. The notion of *étale morphism of ∞-topoi* seems also the correct one. It remains to be seen for which class of morphisms R of ∞-topoi each $f \in R$ can be written as a composition of an étale morphism and a proper morphism.

2.4. **2-categories of spans.** For now, we have not been able to capture in the formalism classes of *étale* and classes *proper* maps, which should have a very important role to play. In order to address this, we will a 2-categorical version of the categories of spans.

Let now be $\mathcal{C}$ be a category with finite limits with L and R as above, but also two new classes, U and D (for *up* and *down*) which satisfy same requirements as the classes L and R, and additionally the *1st out of 3 property.* This is not strictly required for the definition, but it will be for the main use that we will do of this construction, namely in 2.4.1, see below.

Definition 2.4 (1st out of 3)**.** A class M of morphisms which is stable by composition is said to have the *1st out of 3* if for any f and diagram of the form

$$\begin{tikzcd} {} \arrow[r, "f"] \arrow[dr, "M"'] & {} \arrow[d, "M"] \\ & {} \end{tikzcd}$$

then $f \in M$.

The following is an informal description of an $(\infty, 2)$-category as an enriched $(\infty, 1)$-category in $(\infty, 1)$-categories. These $(\infty, 2)$-categories of spans can be formally described as a 2-fold complete Segal anima, see for example [Hau17].

Definition 2.5 (2-category of spans)**.** Let $(\mathcal{C}, L, R, U, D)$ as above. The *2-category of spans* $\mathrm{Span}_{L,R}^{U,D}(\mathcal{C})$ is the $(\infty, 2)$-category with objects the same objects as $\mathcal{C}$ and $(\infty, 1)$-category of morphisms being the full subcategory $\mathrm{Span}_{U,D}(\mathcal{C}_{/(X\times Y)})$ such that morphisms to X are in R and morphisms to Y are in L.

Note that the objects of $\mathrm{Span}_{U,D}(\mathcal{C}_{/(X\times Y)})$, which are equal to objects of $\mathcal{C}_{/(X\times Y)}$, are indeed pair of morphisms

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

The data of a 2-morphism between two correspondences

$$\begin{tikzcd} & Z_1 \arrow[dl, "l_1"'] \arrow[dr, "r_1"] & & & Z_2 \arrow[dl, "l_2"'] \arrow[dr, "r_2"] & \\ X & & Y & X & & Y \end{tikzcd}$$

is then given by a correspondence $Z_1 \xleftarrow{u} Z_3 \xrightarrow{d} Z_2$, with $u \in U$ and $d \in D$ that can be depicted as

$$\begin{tikzcd} & Z_1 \arrow[dl, "l_1"'] \arrow[dr, "r_1"] & \\ X & Z_3 \arrow[u, "u"'] \arrow[d, "d"] & Y \\ & Z_2 \arrow[ul, "l_2"] \arrow[ur, "r_2"'] & \end{tikzcd}$$

with composition being composition of correspondences. Note that if $\mathcal{C}$ is an ordinary category (which is a pertinent case), we just defined a 2-category.

2.4.1. *Encoding adjunctions requirements.* Recall that we are defining this in order to try to add to the formalism classes of maps that behaves as *proper* and *étale* maps. Recall that the philosophy is that

- for a proper map
$$f^* \dashv f_* = f_! \dashv f^!$$
- and for an étale map
$$f_! \dashv f^* = f^! \dashv f_*.$$

The most important input is that this additional 2-categorical data will allow us to formulate when f^* admits as a right adjoint $f_!$ (meaning that $f_* = f_!$, the "proper" case), and dually allows us to formulate when $f_!$ admits f^* as a right adjoint (meaning that $f^* = f^!$, the "étale" case). Suppose that $U \subset R$. The key fact is that in $\mathrm{Span}^{U,\mathrm{iso}}_{\mathrm{All},R}(\mathcal{C})$, for $f : X \to Y$ in R, morphisms

$$f^L\colon \left(\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, equal] & \\ Y & & X \end{tikzcd}\right) \qquad f^R\colon \left(\begin{tikzcd} & X \arrow[dl, equal] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}\right)$$

are adjoint[7]

$$f^L \dashv f^R.$$

The co-unit $f^L \circ f^R \to \mathrm{id}_X$ being the 2-morphism

$$\begin{tikzcd} & X \times_Y X \arrow[dl] \arrow[dr] & \\ X \arrow[dr, equal] & X \arrow[u, "\Delta"] \arrow[d, equal] & X \arrow[dl, equal] \\ & X & \end{tikzcd}$$

[7]There is a general notion of adjuction in a 2-category, see for example [SchSix, Definition 5.8]

and the unit $\mathrm{id}_Y \to f^R \circ f^L$ being the 2-morphism

$$\begin{tikzcd} & Y & \\ Y \arrow[ur, equal] & X \arrow[u, "f"] & Y \arrow[ul, equal] \\ & X \arrow[ul, "f"] \arrow[u, equal] \arrow[ur, "f"'] & \end{tikzcd}$$

Note that we used the *1st out 3 property* to ensure that the diagonal is in U. Recall now that in the formalisation of 6-functor formalisms above, f^L is what ends up being sent to f^* and f_R is sent to $f_!$. What we need to remember is that this new 2-categorical data on correspondences will allows us to force morphisms in the class U to have the property that $f^* = f^!$. So that the class U will end up being what will play the role of proper morphisms.

Note that a dual discussion implies that the class D will play the role of étale morphisms.

Crucial parts of the above explanation is contained in the following proposition.

Proposition 2.6 (Universal property of $\mathrm{Span}^{R,\mathrm{iso}}_{\mathrm{All},R}(\mathcal{C})$. [GR17, Chapter 7, theorem 3.2.2])**.** *Suppose that we are in the context that we made precise at the beginning of the section. Let $\mathcal{D}$ be an $(\infty, 2)$-category. Then the functor*

$$\mathrm{Fun}(\mathrm{Span}^{R,iso}_{All,R}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$$

is fully-faithful at the level of maximal subgroupoids of these functor categories. The image consists of functors such that $\forall f \in R$ we have that f^* has a right adjoint $f_!$, compatible with base change, meaning that for any cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g_X"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g_Y"'] & Y \end{tikzcd}$$

then $g_Y^* f_! \cong f'_! g_X^*$.

The base change result comes from the definition of the composition of correspondences. The universal property with the existence of adjoints comes from the explanation above and the fact $(\infty, 2)-$functors preserves adjunctions.

Remark. There is a dual statement involving *down* morphisms.

2.5. **Enhanced 6-functor formalisms.** In practice, to show the existence of the exceptional functor $f_!$, we use existence of *compactifications*. A compactification for $f\colon X \to Y$ in R is a factorization

$$\begin{tikzcd} X \arrow[r, "j \text{ étale}"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "p \text{ proper}"] \\ & Y \end{tikzcd}$$

The philosophy is that for both étale and proper maps we usually know the existence of this exceptional functor and its desired properties, and a posteriori,

$$f_! \cong p_! j_! = p_* j_!$$

would necessary hold.

In what follows we will change the notation for the class U and denote it by P and call it *proper maps*. Similarly we will change the notation for the class D and denote it by E and call it *étale maps*. The following expected theorem[8] subsumes why this $(\infty, 2)$-category of spans provides the possibility to enhance six-functor formalisms by including étale and proper data.

Expected Theorem 2.7 (Gaitsgory, Rozenblyum)**.** *Let $(\mathcal{C}, R, P, E)$ as above with $P, E \subset R$. Suppose that every morphism in $P \cap E$ is n-truncated for some $n \geq -2$. Assume also that E and P satisfy the 1st out of 3. If for every $f : X \to Y$ in R, the ∞-category*

$$\mathrm{Comp}(f) = \left\{ \begin{tikzcd} X \arrow[r, "j\ \textit{étale}"] \arrow[dr, "f"'] & \overline{X} \arrow[d, "p\ \textit{proper}"] \\ & Y \end{tikzcd} \right\}$$

is non-empty, then for every $(\infty, 2)$-category $\mathcal{D}$ the functor

$$\mathrm{Fun}(\mathrm{Span}_R^{P,E}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$$

is fully-faithful at the level of maximal subgroupoids of these functor categories. The image consists of functors such that

- *for every $p \in P$, p^* has a right adjoint $p_!$, compatible with base change*[9]
- *for every $j \in E$, j^* has a left adjoint $j_!$, compatible with base change,*
- *For every cartesian diagram of the form*

$$\begin{tikzcd} \cdot \arrow[r, "j'"] \arrow[d, "p'"'] & \cdot \arrow[d, "p \in P"] \\ \cdot \arrow[r, "j \in E"'] & \cdot \end{tikzcd}$$

  we have $j'_! p_! \cong p'_! j_!$.

Remark. One can compare the assumptions of the above theorem on classes E and P with Mann's notion of *suitable decomposition* [Man22, Definition A.5.9].

We can now define a notion of six-functor formalism who takes into account proper and étale morphisms.

Definition 2.8 (Enhanced 3-functor formalism)**.** Let $(\mathcal{C}, R, P, E)$ with the same hypothesis as in the Expected Theorem 2.7. An *enhanced 3-functor formalism* is a right-lax symmetric monoidal functor

$$\mathrm{Span}_R^{P,E}(\mathcal{C}) \to \mathbf{Cat}_\infty$$

where $\mathbf{Cat}_\infty$ denotes the $(\infty, 2)$-category of $(\infty, 1)$-categories.

[8] The expected theorem 2.7 relies on some $(\infty, 2)$-categorical facts that still need to be proven. Partials versions of 2.7 is [GR17, Chapter 7, Theorem 4.1.3, Theorem 5.2.4]. See also [LH, Remark 4.2.5]

[9] So in this case we may write $p_! = p_*$

Definition 2.9 (Enhanced 6-functor formalism)**.** Let $(\mathcal{C}, R, P, E)$ with the same hypothesis as above. An *enhanced 6-functor formalism* is a enhanced 3-functor formalism

$$\mathcal{D}\colon \operatorname{Span}_R^{P,E}(\mathcal{C}) \to \mathbf{Cat}_\infty$$

such that

(1) for each object $X \in \mathcal{C}$ the symmetric monoidal category $\mathcal{D}(X)$ is closed,
(2) for each map $f : X \to Y$, the functor $f^* : \mathcal{D}(Y) \to \mathcal{D}(X)$ is a left adjoint and
(3) for each map $f : X \to Y$, the functor $f_! : \mathcal{D}(X) \to \mathcal{D}(Y)$ is a left adjoint.

Note that we can replace (3) by

(3)' for each map $f : X \to Y$ in P, the functor $f_! : \mathcal{D}(X) \to \mathcal{D}(Y)$ is a left adjoint.

Indeed if $f = p \circ j$ with $p \in P$ and $j \in E$, then $f^! = j^* p^!$.

2.5.1. *Separated and unramified maps.* Suppose that $\mathcal{D}$ is an enhanced six-functor formalism on $(\mathcal{C}, R, P, E)$. Define the class S (for separated) of morphisms $X \to Y$ in $\mathcal{C}$ such that the diagonal $\Delta : X \to X \times_Y X$ is in P. If P are proper morphisms in the context of topological spaces or schemes, this corresponds to the right notion of separated morphisms. Then the following two cell

$$\begin{tikzcd} & X \times_Y X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & X \arrow[u, "\Delta"] \arrow[d, equal] & X \\ & X \arrow[ul, equal] \arrow[ur, equal] & \end{tikzcd}$$

gives a natural 2-morphism $f^* f_! \cong p_{2,!} p_1^* \to \mathrm{id}$, (where we used base change formula) so by adjunction we get a natural morphism

$$f_! \to f_*.$$

Note that if f is proper, then the above 2-morphism is the co-unit of an adjunction, the unit being (this is the same as in section 2.4.1)

$$\begin{tikzcd} & Y \arrow[dl, equal] \arrow[dr, equal] & \\ Y & X \arrow[u, "f"] \arrow[d, equal] & Y \\ & X \arrow[ul, "f"] \arrow[ur, "f"'] & \end{tikzcd}$$

implying that $f_!$ is the right adjoint to f^* and that the above constructed $f_! \to f_*$ map is an isomorphism.

Dually we may define a class U (this time for *unramified*) of morphisms such that the diagonal is étale. In this case we have a natural morphism $\mathrm{id} \to f_* f^!$ so a by adjunction a natural morphism, which will be an isomorphism if the map is étale,

$$f^* \to f^!.$$

In the context of topological spaces, if the diagonal of $X \to Y$ is open, this exactly means that for every $x \in X$ there is an open neighbourhood U of x with $f_{|U}$ injective, so a local injection. In algebraic geometry this coincide with the usual notion of unramified morphism.

2.5.2. *Open and closed maps.* Note that given a context $(\mathcal{C}, R, P, E)$ as above it is legitimate to define two classes C (for closed) and O (for open) of morphisms as respectively the classes of monomorphisms in P and E. Again in the context of topological spaces or schemes it coincides with the usual notions of closed and open immersions.

2.5.3. *Functorialites of compact support cohomology and Borel–Moore homology.* The additionnal data of the two-cells relating correspondences with proper and étale maps allows to encode the bivariant functoriality of compact support cohomology and Borel–Moore homology. Let

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & X' \arrow[d, "h"] \\ & Y \end{tikzcd}$$

be a commutative diagram. We see the following.

(1) Compact support cohomology is *contravariant* with respect to proper maps. If f is proper, we have

$$h_!h^* \to g_!g^* = h_!f_!f^*h^* = h_!f_*f^*h^*$$

using the unit $1 \mapsto f_*f^*$. This also summarized in the data of the following two-cell.

$$\begin{tikzcd} & X' \arrow[dl, "h"'] \arrow[dr, "h"] & \\ Y & X \arrow[u, "f"] \arrow[d, equal] & Y \\ & X \arrow[ul, "g"] \arrow[ur, "g"'] & \end{tikzcd}$$

(2) Compact support cohomology is *covariant* with respect to étale maps. If f is étale, we have

$$g_!g^* = h_!f_!f^*h^* = h_!f_!f^!h^* \to h_!h^*$$

using the counit $f_!f^! \mapsto 1$. This also summarized in the data of the following two-cell.

$$\begin{tikzcd} & X \arrow[dl, "g"'] \arrow[dr, "g"] \arrow[d, equal] & \\ Y & X \arrow[d, "f"] & Y \\ & X' \arrow[ul, "h"] \arrow[ur, "h"'] & \end{tikzcd}$$

The following is dual.

(3) Borel–Moore homology is *covariant* with respect to proper maps. If f is proper we have
$$g_*g^! = h_*f_*f^!h^! = h_*f_!f^*h^* \to h_*h^*$$
using the counit $f_!f^! \to 1$.

(4) Borel–Moore homology is *contravariant* with respect to étale maps. If f is étale we have
$$h_*h^! \to g_*g^! = h_*f_*f^!h^! = h_*f_*f^*h^!$$
using the unit $1 \mapsto f_*f^*$.

Remark. Note that if we define *finite étale* to be *proper and étale*, then these (co)hmological notions are *both* covariant and contravariant with respect to finite étale maps. Note that in the topological case a map is finite étale if and only if it is a finite covering.

We also see, still in the topological realm say, that there is an exceptional covariant functoriality of cohomology for a local homeomorphism between compact spaces, and dually an exceptional contravariance for homology in the same case.

3. Motivic spectra and motivic six-functor formalism for schemes

In this lecture, we will introduce motivic spectra. References are Morel and Voevodsky [MV99] and also Cisinski and Déglise [CD19]. We will also explain how these motives are coefficients for an universal (in a sense that we will make precise) cohomology theory for schemes in the context of a general formalism of six-functor formalism for schemes. This was exposed by Ayoub in [Ayo07a] and [Ayo07b].

The formalism of six operations was first envisioned by Grothendieck in the context of l-adic étale cohomology for schemes as a way to address Weil conjectures. Namely the étale analogue of the Lefschetz trace formula and Poincaré duality would directly respectively imply the first (rationality) and the second one (functional equation). The key input of Grothendieck was that the Zeta-function of Weil conjectures was a shadow of the behaviour of cohomology theories for schemes.

In algebraic geometry, there are many cohomology theories, l-adic ones, one for each l, crystalline cohomology, de Rham cohomology... Grothendieck envisioned universal coefficients which would dictate every other cohomology theory for schemes. Namely, Grothendieck's idea was that of an initial functor $\mathrm{Mot}\colon \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Mot}(k)$, (the associated motive) where $\mathrm{Mot}(k)$ should be an abelian category, such that for any "cohomology functor" $h^* : \mathrm{Sch}_k \to \mathbb{A}$ for an abelian category $\mathbb{A}$, the there should be a factorization

$$\begin{tikzcd}
\mathrm{Sch}_k^{\mathrm{op}} \arrow[r, "h^*"] \arrow[dr, "\mathrm{Mot}"'] & \mathbb{A} \\
& \mathrm{Mot}_k \arrow[u, dashed]
\end{tikzcd}$$

Motivic cohomology is the name of the Ext-groups from this conjectural abelian category of motives. Even though Grothendieck saw that some results on these conjectural motives could imply the last and most difficult part of Weil conjectures, he was more so interested by the development of the theory of motives themselves which he saw as "his most profound contribution to the mathematics of his time" [Gro22, p. 156].

With $\mathbb{A}^1$-homotopy theory, Morel and Voevodsky [MV99] introduced the so called derived category of *motivic spectra*, which is close in spirit to the initial character of motives described above. Namely the association $X \mapsto \mathrm{MSp}(X)$ extend to a six-functor formalism which is *initial* with respect to all six-functor formalisms which satisfy a set of axioms that we will explain.

Motivic spectra should be universal coefficients for cohomology theories on schemes, a role played by shaves of spectra in the context of topological spaces. Here are some analogies.

Topological spaces	Schemes
Proper/smooth/étale	Proper/smooth/étale
locally proper	locally of finite type=lft
$\mathrm{Sh}(X, \mathrm{Sp})$	Motivic spectra $\mathrm{MSp}(X)$

3.1. **Six-functor formalisms for schemes–Ayoub's axioms.** We first present the following result due to Ayoub. Axioms mentioned in the theorem below will be explained in the rest of this subsection.

Theorem 3.1 (Ayoub, after Voevodsky, Röndigs)**.** *Let*

$$\mathcal{D}\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Pr}^L_\infty)$$

satisfying Ayoub's axioms *(see below). Then $\mathcal{D}$ extends uniquely to an enhanced 6-functor formalism*

$$\mathcal{D}\colon \mathrm{Span}^{\textit{proper, étale}}_{\textit{lft}} \to \mathrm{Pr}^L_\infty$$

in which Atiyah duality holds; namely there exist a symmetric monoidal functor[10]

$$(K(X), +) \to (\mathrm{Pic}(\mathcal{D}(X)), \otimes)$$

such that if $f : X \to Y$ is smooth then the natural map

$$\omega_f \otimes f^* \to f^!$$

is an isomorphism, with $\omega_f \coloneqq f^!\mathbb{1} \cong S^{T_f}$. Here T_f is the relative tangent bundle $\Omega^\vee_{X|Y}$.

3.1.1. *Ayoub's axioms.* We list now the axioms mentionned in the above theorem. These axioms are meant to formulate what should be a cohomology theory for schemes in the language of six-operations.

(1) *Zariski descent.* The functor $\mathcal{D}$ is a sheaf for the Zariski topology on Sch.
(2) *Smooth base change.* For every smooth map $f\colon X \to Y$, the functor f^* has a left adjoint $f_\sharp$ such that for all cartesian squares

$$\begin{tikzcd} X' \arrow[r, "g_X"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g_Y"'] & Y \end{tikzcd}$$

the natural map $g_Y^* f_\sharp \to f'_\sharp g_X^*$ is an isomorphism and a smooth projection formula, for $A \in \mathcal{D}(Y)$ and $B \in \mathcal{D}(X)$ the natural map

$$A \otimes f_\sharp B \to f_\sharp(f^*A \otimes B)$$

is an isomorphism.
(3) *Localization.* Let $\iota\colon Z \to X$ a closed embedding with open complement $j\colon U \to X$. Then (ι^*, j^*) is conservative and ι_*, j_* are fully faithful.
(4) $\mathbb{A}^1$*-homotopy invariance.* Let

$$p\colon X \times \mathbb{A}^1 \to X$$

the projection. Then p^* is fully-faithful.
(5) $\mathbb{P}^1$*-stability.* Let $s\colon X \to X \times \mathbb{A}^1$ the zero section. Then

$$p_\sharp s_*\colon \mathcal{D}(X) \to \mathcal{D}(X)$$

is an equivalence.

[10]This functor should be thought as sending a vector bundle to the associate sphere bundle.

Remark. The localization amounts to a *recollement* if the functor $\mathcal{D}$ takes value in stable ∞-categories.

Remark. We explain what is the functor $f_\sharp$ *a posteriori* (supposing $\mathcal{D}$ fits in a 6-functor formalism). For a smooth morphism $f : X \to Y$,

(a) Poincaré duality should hold, meaning $f^! \cong \omega_f \otimes f^*$,
(b) the dualizing complex ω_f should be invertible and
(c) the dualizing complex should respect base change.

Note that then $f^* \cong f^! \otimes \omega_f^{-1}$. And for $A \in \mathcal{D}(X)$ and $B \in \mathcal{D}(Y)$

$$\mathrm{Map}_{\mathcal{D}(X)}(A, f^! B \otimes \omega_f^{-1}) \cong \mathrm{Map}_{\mathcal{D}(X)}(A \otimes \omega_f, f^! B) \cong \mathrm{Map}_{\mathcal{D}(Y)}(f_!(A \otimes \omega_f), B).$$

In other words, $f_\sharp = f_!(- \otimes \omega_f)$. Now, one sees that under listed above three assumptions on smooth morphisms[11] the base change formula and the projection formula holds for $f_\sharp$ if and only if it holds for $f_!$.

Remark. It is unclear for now why the fifth axiom is called $\mathbb{P}^1$-*stability*. We will see that it actually implies in the case of motivic spectra that $\mathbb{P}^1_X$ is invertible with respect to $\otimes$, which will imply the *stability* of the category. This is an analogue in the topological case to the fact the S^1 is invertible in $(\mathrm{Sp}, \otimes)$, which amounts to Σ being an equivalence, so *stability*.

Here's an existence theorem about the initial six-functor formalism for schemes who satisfies Ayoub's axioms.

Theorem 3.2 (Drew–Gallauer)**.** *The ∞-category of functors $\mathcal{D}$'s satsyfing Ayoub's axioms has an inital object* MSp*, called* motivic spectra*.*

We will in the rest of the lecture explain how to construct this category $\mathrm{MSp}(X)$ for a scheme X.

3.2. **Construction of motivic spectra.** In what follows we denote by Sm_X the category of smooth X-schemes.

The following topology play a key role in the construction of motivic spectra.

Definition 3.3 (Nisnevich topology)**.** Let X be a scheme. A familly of étale maps $(U_i \to X)$ is called a Nisnevich cover if for all fields k

$$\coprod_i U_i(k) \to X(k)$$

is surjective. We call the *Nisnevich topology* the topology on Sm_X generated by those families of maps.

Remark. Nisnevich topology is finer than Zariski topology but coarser than the étale topology. In the étale topology we ask the map to be surjective only for all *algebraically closed* fields allowing for more maps to be covers. We mention that there is a neat interpretation of the Nisnevich topology in term of the universal topology with respect to *Henselian local rings.* Namely, localization at points of rings of functions in the Nisnevich topology are Henselian local rings. Henselian local rings are local rings for which Hensel's lemma holds. See [Stacks, Tag 04GE] for more on Henselian local rings.

[11] On this subject, see the notion of a cohomologically smooth morphism in [SchSix, Definition 5.1].

Definition 3.4 ($\mathbb{A}^1$-invariant Nisnevich sheaves)**.** A presheaf

$$F\colon \mathrm{Sm}_X^{\mathrm{op}} \to \mathrm{Ani}$$

is called a $\mathbb{A}^1$-*invariant Nisnevich sheaf* if for all Nisnevich covers $(U_i \to U)$ in Sm_X the natural map

$$F(U) \xrightarrow{\sim} \varprojlim \big(\prod F(U_i) \rightrightarrows \prod F(U_i \times_U U_j) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \cdots \big)$$

is an isomorphism and $F(U) \xrightarrow{\sim} F(U \times \mathbb{A}^1)$.

We denote by

$$\mathrm{MAni}(X) = \{\mathbb{A}^1 - \text{invariant Nisnevich sheaves}\} \subset \mathrm{Fun}(\mathrm{Sm}_X^{\mathrm{op}}, \mathrm{Ani}).$$

This inclusion is a right adjoint.

We can now state the following construction of *motivic spectra.*

Theorem 3.5 (Construction of Motivic spectra)**.** *The stable ∞-category of motivic spectra can be constrcuted as*

$$\mathrm{MSp}(X) = \mathrm{MAni}(X)_*[(\mathbb{P}^1_X/\infty)^{\otimes -1}]$$

meaning that we inverted the relative projective line pointed at infinity $\mathbb{P}^1_X/\infty$ with respect to the tensor product.

Remark. The stability will be explained below, when we will address why this construction satisfy Ayoub axiom (5) $\mathbb{P}^1$-*stability*.

Remark. We point out in this remark that this construction is actually an analogue of a topological construction of sheaves of spectra. Namely, let X be now a topological space and Sm_X the category of manifolds bundles of X. Equip Sm_X of the topology generated by open coverings. In the definition of homotopy invariance replace $- \times \mathbb{A}^1$ by $- \times \mathbb{R}$. Now, we have

$$\mathrm{Sh}(X, \mathrm{Sp}) = \{\text{Homotopy invariant sheaves on } \mathrm{Sm}_X\}[(S^1)^{\otimes -1}].$$

We will now explain why with this construction $\mathrm{MSp}(X)$ satisfy Ayoub's axioms.

(1) *Zariski descent.* $X \mapsto \mathrm{MSp}(X)$ is even a Nisnevich sheaf by construction; as Nisnevich topology is finer than the Zariski one, axiom (1) follows.

(2) *Smooth base change.* One can show that the forgetful functor adjoint to the pullback

$$\mathrm{Sm}_Y \underset{f^*}{\overset{f_\sharp}{\leftrightarrows}} \mathrm{Sm}_X$$

will transport through the construction.

(4) $\mathbb{A}^1$-*homotopy invariance.* We want to show that

$$p^*\colon \mathrm{MSp}(X) \to \mathrm{MSp}(X \times \mathbb{A}^1)$$

is fully faithful. This holds if and only if the co-unit map $p_\sharp p^* \to \mathrm{id}$ is an isomorphism. But

$$p_\sharp p^* U = U \times \mathbb{A}^1 \to U$$

is an isomorphism as we forced $\mathbb{A}^1$-invariance.

(5) $\mathbb{P}^1$-*stability.* Let us fix the following notations, with s being the zero section, and $\mathbb{A}^1 \setminus 0$ denoting the scheme $\mathbb{G}_m$. We choose this notation because the following calculations borrow good intuition from the "real" case.

$$X \times \mathbb{A}^1 \setminus 0 \xrightarrow{j} X \times \mathbb{A}^1 \underset{p}{\overset{s}{\leftrightarrows}} X$$

We first remark that

$$p_\sharp s_*(\mathbb{1}) \otimes - \cong p_\sharp s_*(\mathbb{1} \otimes s^* p^*) \cong p_\sharp s_*,$$

because both $p_\sharp$ and s_* satisfy projection formula (the latter is proper being a closed immersion) and $ps = \mathrm{id}$. In other words, the functor $p_\sharp s_*$ is an equivalence if and only if $p_\sharp s_*(\mathbb{1})$ is $\otimes$-invertible.

To show this, we will use the following consequence of the *localization axiom* (that we will address next), namely that

$$j_\sharp j^* \to \mathrm{id} \to s_* s^*$$

is a cofiber sequence. Therefore $p_\sharp s_*(\mathbb{1}) = p_\sharp s_*(s^* \mathbb{1})$ is equal to

$$\mathrm{cofib}(p_\sharp j_\sharp j^* \mathbb{1} \to p_\sharp \mathbb{1}).$$

But the latter is the cofiber

$$\mathbb{A}^1 \setminus 0 \to \mathbb{A}^1 \to \mathbb{A}^1/(\mathbb{A}^1 \setminus 0).$$

Now we claim that the bottow arrows of the following diagram, where vertical arrows are cofiber sequences, are equivalences.

$$\begin{tikzcd}
\mathbb{A}^1 \setminus 0 \arrow[r] \arrow[d] & \mathbb{P}^1 \setminus 0 \cong \mathbb{A}^1 \arrow[d] & * \arrow[l, "\sim"'] \arrow[d, "\infty"] \\
\mathbb{A}^1 \arrow[r] \arrow[d] & \mathbb{P}^1 \arrow[r, equal] \arrow[d] & \mathbb{P}^1 \arrow[d] \\
\mathbb{A}^1/(\mathbb{A}^1 \setminus 0) \arrow[r] & \mathbb{P}^1/(\mathbb{P}^1 \setminus 0) & \mathbb{P}^1/\infty \arrow[l]
\end{tikzcd}$$

The fact the right bottom arrow is an equivalence follows from the fact that the top right arrow is an equivalence by $\mathbb{A}^1$-invariance. Note also that the left upper square is a pushout of Zarsiki sheaves, namely this is the standard gluing of $\mathbb{P}^1$. It then follows that the left bottow arrow is an equivalence.

Now, in the construction of motivic spectra we inverted $\mathbb{P}^1$ with respect to $\otimes$, so $\mathbb{A}^1/(\mathbb{A}^1 \setminus 0)$ is $\otimes$-invertible has required.

With this explanation, we also understood why this axiom is called $\mathbb{P}^1$-*stability.*

Remark. Note that the upper left pushout square

$$\begin{tikzcd}
\mathbb{A}^1 \setminus 0 = \mathbb{G}_m \arrow[r] \arrow[d] & \mathbb{P}^1 \setminus 0 \cong \mathbb{A}^1 \arrow[d] \\
\mathbb{A}^1 \arrow[r] & \mathbb{P}^1
\end{tikzcd}$$

also implies that $\Sigma\mathbb{G}_m = \mathbb{P}^1$ using $\mathbb{A}^1 \simeq *$. We inform that Morel–Voevodsky [MV99, p. 111] have defined *motivic spheres* as

$$S^{p,q} := \Sigma^{p-q}\mathbb{G}_m^{\otimes q}.$$

Note also then that

$$S^1 \otimes \mathbb{G}_m \otimes (\mathbb{P}^1)^{\otimes -1} = \mathbb{P}^1 \otimes (\mathbb{P}^1)^{\otimes -1} = \mathbb{1}$$

implying that S^1 and $\mathbb{G}_m$ are both $\otimes$-invertible. In particular it follows that $\mathrm{MSp}(X)$ is *stable*, and that all motivic spheres are $\otimes$-invertible.

We now address the most difficult axiom to show, namely the *localization axiom.* Note also that the use of the Nisnevich topology was not shown yet. It becomes crucial here.

(1) *Localization axiom.* This is a consequence of the following theorem [MV99, pp. 113–118].

Theorem 3.6 (Morel–Voevodsky)**.** *Let $\iota\colon Z \to X$ a closed subscheme with open complement $j\colon U \to X$ then*

$$\begin{tikzcd} \mathrm{MAni}(Z) \arrow[r, "\iota_*"] \arrow[d] & \mathrm{MAni}(X) \arrow[d, "j^*"] \\ * \arrow[r] & \mathrm{MAni}(U) \end{tikzcd}$$

is a cartesian square of ∞-categories.

This theorem uses *all aspects* of the definitions,
- generated by smooth schemes,
- the Nisnevich topology,
- $\mathbb{A}^1$-invariance.

The Henselian property of local rings in the Nisnevich topology[12] is crucially used as well as a version of a tubular neighbourhood theorem in algebraic geometry. If $Z \to X$ is closed then there exists a scheme V (to be thought as a tubular neighbourhood) with

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d] & Z \arrow[r, equal] \arrow[d] & Z \arrow[d, "\text{zero section}"] \\ X & V \arrow[l, "\text{étale}"'] \arrow[r, "\text{étale}"'] & N_{Z/X} \end{tikzcd}$$

Here are some more results that highlight the necessity of the Nisnevich topology.

Proposition 3.7 (Morel–Voevodsky, characterization of Nisnevich descent)**.** *Let $\mathcal{C}$ be a complete ∞-category, and $F : \mathrm{Sch}^{\mathrm{op}} \to \mathcal{C}$. The following are equivalent.*

(1) F is a Nisnevich sheaf.
(2) F is a Zariski sheaf and satisfies Nisnevich excision.

[12] Morel and Voevodsky clearly put the emphasis that this result would not hold if Zariski topology was used to construct motivic spectra [MV99, p. 113].

Nisnevich excision *being defined as sending every cartesian square of the form*

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d, "p \text{ étale, iso over } X\setminus U"] \\ U \arrow[r, "open"'] & X \end{tikzcd}$$

to a cartesian square.

Proposition 3.8. *If $\mathcal{D} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Cat}_\infty^{stable}$ satisfies smooth base change and localization, then $\mathcal{D}$ satisfy Nisnevich excision.*

In particular, we see from these last two results that it is a consequence of Ayoub's axioms that $\mathcal{D}$ is necessarily a Nisnevich sheaf.

Actually, from this last fact, Theorem 3.6 is a formal consequence. This is explained in [DG22].

3.3. **Concluding remarks.** We mention that finding a 6-functor formalism for schemes *without $\mathbb{A}^1$-homotopy invariance* is a major open problem. Such a formalism should capture all known cohomology theories as K-theory, THH, TC, syntomic cohomology, prismatic cohomology, de Rham chomology, ... Recent works [EM23],[AHI24a],[AHI24b] goes into this direction.

All these cohomologies satisfy Nisnevich descent and $\mathbb{P}^1$-stability but not localization and $\mathbb{A}^1$-invariance. The localization axiom should be replaced by a *formal* (in the "formal scheme" sense) variant.

References

[AHI24a] Toni Annala, Marc Hoyois, and Ryomei Iwasa. *Algebraic cobordism and a Conner-Floyd isomorphism for algebraic K-theory*. 2024. arXiv: 2303.02051 [math.AG].

[AHI24b] Toni Annala, Marc Hoyois, and Ryomei Iwasa. *Atiyah duality for motivic spectra*. 2024. arXiv: 2403.01561 [math.AG].

[Ayo07a] Joseph Ayoub. *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique (I)*. fr. Astérisque 314. Société mathématique de France, 2007. URL: http://www.numdam.org/item/AST_2007__314__R1_0/.

[Ayo07b] Joseph Ayoub. *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique (II)*. fr. Astérisque 315. Société mathématique de France, 2007. URL: http://www.numdam.org/item/AST_2007__315__1_0/.

[CD19] Denis-Charles Cisinski and Frédéric Déglise. *Triangulated Categories of Mixed Motives*. Springer International Publishing, 2019. DOI: 10.1007/978-3-030-33242-6. URL: https://doi.org/10.1007%2F978-3-030-33242-6.

[DG22] Brad Drew and Martin Gallauer. "The universal six-functor formalism". In: *Annals of K-Theory* 7.4 (2022), pp. 599–649. DOI: 10.2140/akt.2022.7.599. URL: https://doi.org/10.2140%2Fakt.2022.7.599.

[EM23] Elden Elmanto and Matthew Morrow. *Motivic cohomology of equicharacteristic schemes*. 2023. arXiv: 2309.08463 [math.KT].

[GR17] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry. Vol. I. Correspondences and duality*. Vol. 221. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2017, pp. xl+533. ISBN: 978-1-4704-3569-1. DOI: 10.1090/surv/221.1. URL: https://doi.org/10.1090/surv/221.1.

[Gro22] Alexandre Grothendieck. In: *Récoltes et semailles: Réflexions et témoignage d'un passé de mathématicien*. Gallimard, 2022.

[HA] Jacob Lurie. *Higher Algebra*. 2017. URL: https://people.math.harvard.edu/~lurie/papers/HA.pdf.

[Hat02] Allen Hatcher. "Section 3.3 Poincaré Duality, Orientations and Homology". In: *Algebraic topology*. Cambridge University Press, 2002.

[Hau17] Rune Haugseng. "Iterated spans and classical topological field theories". In: *Mathematische Zeitschrift* 289.3-4 (2017), pp. 1427–1488. DOI: 10.1007/s00209-017-2005-x. URL: https://doi.org/10.1007%2Fs00209-017-2005-x.

[Hoy17] Marc Hoyois. *Higher Galois theory*. 2017. arXiv: 1506.07155 [math.CT].

[HTT] Jacob Lurie. *Higher Topos Theory*. Annals of mathematics studies. Princeton, NJ: Princeton University Press, 2009.

[Ker] Jacob Lurie. *Kerodon*. https://kerodon.net. 2023.

[LH] Jabob Lurie and Michael Hopkins. *Ambidexterity in $K(n)$-Local Stable Homotopy Theory*. URL: https://people.math.harvard.edu/~lurie/papers/Ambidexterity.pdf.

[LuDmod] Jacob Lurie. *Notes on Crystals and algebraic D-modules*. URL: https://ncatlab.org/nlab/files/Lurie-NotesOnCrystals.pdf.

[Man22] Lucas Mann. *A p-Adic 6-Functor Formalism in Rigid-Analytic Geometry*. 2022. arXiv: 2206.02022 [math.AG].

[MV99] Fabien Morel and Vladimir Voevodsky. "A^1-homotopy theory of schemes". en. In: *Publications Mathématiques de l'IHÉS* 90 (1999), pp. 45–143. URL: http://www.numdam.org/item/PMIHES_1999__90__45_0/.

[PF-G] Bhargav Bhatt. *Lecture (Fall 2022): Prismatic F-gauges*. URL: https://www.math.ias.edu/~bhatt/teaching/mat549f22/.

[SchSix] Peter Scholze. *Lecture (Winter 2022/23): Six-functor formalisms*. URL: https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf.

[SGA4] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier. *Theorie de Topos et Cohomologie Etale des Schemas I, II, III*. Vol. 269, 270, 305. Lecture Notes in Mathematics. Springer, 1971.

[Stacks] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu. 2023.

[Vol23a] Marco Volpe. *The six operations in topology*. 2023. arXiv: 2110.10212 [math.AT].

[Vol23b] Marco Volpe. *Verdier duality on conically smooth stratified spaces*. 2023. arXiv: 2206.02728 [math.AT]. URL: https://arxiv.org/abs/2206.02728.