MODULI FUNCTORS AND AUTOMORPHISMS

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Abstract. This note is meant to analyze precisely how "automorphisms are an obstruction to the representability of moduli". In particular we define precisely what is a moduli functor.

1. Introduction

We make a precise definition of "moduli functor".

Definition 1.1. Let $(\mathcal{C}, \tau)$ a category equipped with a subcanonical Grothendieck topology. A *moduli functor* $\mathcal{M}\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ani}_{\leq 1}$ is a substack of $S \mapsto \mathcal{C}^{\simeq}_{/S}$.

Remark. This definition makes sense for a n-category $\mathcal{C}$ for $0 \leq n \leq \infty$ and then the natural notion of moduli functor lands in $\mathrm{Ani}_{\leq n}$.

In [HTT, p. 6.1.6] Lurie shows that any moduli functor with value in Ani on an ∞-topos is representable. This is simply that sheaves on a topos are representable which can be viewed as a tautology.

This is the quick definition where the word "stack" encompasses some data:

(1) for each object S of $\mathcal{C}$ the set $\pi_0(\mathcal{M}(S))$ is a set of isomorphism classes of objects over S, and these collections are stable by pullback,
(2) maps $\mathcal{M}(S)(X, Y)$ between two objects $X, Y \in \mathcal{M}(S)$ are a subset of $\mathrm{Iso}_S(X, Y)$ isomorphisms of S-objects. The collection of maps $\mathcal{M}(S)(X, Y)$ is stable under composition and pullback,
(3) we can glue objects and maps in $\mathcal{M}$ with respect to the topology τ.

We will write $M = \pi_0(\mathcal{M})\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, with *no sheafification* considered.

We begin by some computations of first cohomology in order to study more precisely the saying that "automorphisms are obstruction to representability of moduli". The key aspect should be that it allows to construct locally trivial but not globally trivial objects, and take as main example the *Möbius band construction* .

We will study as a toy case the context of topological spaces where the Möbius band idea is here without analogy.

In the case of schemes, we consider

$$C := V_+(x) \cup V_+(y) \cup V_+(z) \subset \mathbb{P}^2_R,$$

where R being any ring such that it's reduction is integral. This is the "triangle" in $\mathbb{P}^2_R$. Denote by $U_x = D_+(x) \cap C$ and similarly for U_y and U_z. These three opens are isomorphic to the cross $X = \mathrm{Spec}(R[st]/(st))$. The goal is to show that C can play the role of the circle where to perform "Möbius band construction" but in the realm of schemes.

We will also provide a meta proposition using the general context introduced above.

2. Computations in first cohomology

We will need some preliminary lemmas. In what follows $\mathcal{C}$ is a category equipped with a Grothendieck topology. We recall that the notion of Čech cohomology (See for example [Sta, Tag 03AK]) is the correct one for H^0 and H^1.[1] For $i = 0, 1$, when we write $\mathrm{H}^i(X, G)$ for an object $X \in \mathcal{C}$ and G a group in Set we mean the Čech cohomology $\mathrm{H}^i(X, \underline{G})$ where $\underline{G}$ denotes the constant sheaf on $\mathcal{C}$ with value G where G can be a group for $i = 1$ or more generally a set for $i = 0$.

Definition 2.1. Let $\mathcal{C}$ be a category equipped with a Grothendieck topology. We say that $U \in \mathcal{C}$ is *connected* if the natural map $S \to \mathrm{H}^0(U, S)$, where S is any set is an isomorphism.

Remark. This is equivalent to saying that there is no two non-empty sub-sheaves U_1 and U_2 of U in $\mathrm{Sh}(\mathcal{C})$ such that $U = U_1 \sqcup U_2$.

The following is a proof by hand that can be useful in many situations to compute first cohomology groups.

Lemma 2.2. *Let $\mathcal{C}$ be a category with a Grothendieck topology with fiber products. Let $X \in \mathcal{C}$ be an object and G a (not necessarily abelian) group. Let $\mathcal{U} = (U_i \to X)$ be any cover of X such that any U_i's and $U_{ij} = U_i \times_X U_j$'s and $U_{ijk} = U_i \times_X U_j \times_X U_k$'s are connected (in particular non-empty). Then*

$$\mathrm{H}^1(\mathcal{U}, G) = 1$$

More precisely, if X is an object in $\mathcal{C}$ and $(U_i)_{i\in I}$ any open cover of X and (φ_{ij}) cocyles, then if $(U_i)_{i\in I_0} = \mathcal{U}$ is a subset of this cover (who does not necessarily cover X) that satisfies the above, then we can find cocycles (φ'_{ij}) cohomologous to (φ_{ij}) such that $\varphi'_{ij} = 1$ for all $i, j \in I_0$ and $\varphi_{ij} = \varphi'_{ij}$ when both i and j are not in I_0.

Proof. Note that the connectedness hypothesis imply that sections on U_i, U_{ij} and U_{ijk} take all value in G. This is implicit in our notation in what follows.

Here is a “by hand” argument. Let's write the set of indices $I_0 = I'_0 \cup j_0$. Let $h_{i_0} = \varphi^{-1}_{i_0 j_0} = \varphi_{j_0 i_0}$ for any $i_0 \in I'_0$ and $h_i = 1$ if $i \notin I'_0$. Denote by δ_i the indicator function of I_0. Therefore $h_i = \varphi^{\delta_i}_{j_0 i}$. Now by construction (φ'_{ij}) where,

$$\varphi'_{ij} = h_i \varphi_{ij} h_j^{-1} = \varphi^{\delta_i}_{j_0 i} \varphi_{ij} \varphi^{\delta_j}_{j j_0}$$

is a cocyle cohomologous to the initial one. As we supposed that triple intersections are not empty for any $i, j, k \in I_0$, the cocycle condition secures the proof of the lemma, in the sense that it shows that indeed (φ'_{ij}) are cocycles, meaning that they also satisfy the cocycle condition.

Another proof. The 2-truncated Čech nerve of $\mathcal{U}$ is the same as the 2-truncated Čech nerve of a complete graph where any triangle in the graph is filled. The latter is contractible therefore $\mathrm{H}^1(\mathcal{U}, G) = 1$. □

Remark. Any irreducible topological space Y satisfies Lemma 2.2 – it implies $\mathrm{H}^1(Y, G) = 0$.

[1] By *correct* we mean that if we look at the $\mathrm{Sh}_{\mathrm{Set}}(\mathcal{C})_{/X}$ and $\pi\colon \mathrm{Sh}(\mathcal{C})_{/X} \to \mathrm{Set}$ the unique morphism, then $\mathrm{H}^0(X, \pi^* S)$ in the Čech sense is $\pi_* \pi^* S$ and $\mathrm{H}^1(X, G)$ in the Čech sense computes isomorphism classes of $\pi^* G$-torsors in $\mathrm{Sh}_{\mathrm{Set}}(\mathcal{C})_{/X}$.

Remark. The triple intersection requirement is really necessary and is at the core of the argument! Indeed the real circle has non trivial first cohomology but is covered with three opens U_i, U_j and U_{j_0} with double intersections being connected and with an empty triple intersection.

We introduce the following.

Definition 2.3. A *1-cohomological circle* in a category $\mathcal{C}$ with a Grothendieck topology τ is an object C with a covering $(U_i \to C)$, by connected objects, with intersection being connected and triple intersection empty with the property that $\mathrm{H}^1_\tau(U_i, G) = 0$ for any constant group G.

Example 2.4. The circle in Top is an example. If R is a ring with an integral reduction, then $\mathrm{Proj}(R[x,y,z]/(xyz))$ is also in Sch_R with the Zariski topology.

Lemma 2.5. *Let $\mathcal{C}$ be a category equipped with a Grothendieck topology τ and C a 1-cohomological circle. If* $\mathrm{Conj}(G)$ *denote conjugacy classes of G we have*

$$\mathrm{H}^1_\tau(C, G) = \mathrm{Conj}(G)$$

Any torsor is given by the "Möbius band construction".

Proof. We compute the Čech cohomology with the aforementionned cover with coefficient in a constant group G. Note that there is no cocycle condition as the triple intersection is empty. We can apply the lemma to the subset of the cover U_i and U_j to get only coycles of the form $(1,1,g)$. *That's what we mean by a torsor obtained by the "Möbius band construction".* Now we see that $(1,1,g)$ and $(1,1,g')$ with $g, g' \in G$ are cohomologous if and only g and g' are conjugate to each other. Therefore

$$\mathrm{H}^1_\tau(C, G) = \mathrm{Conj}(G)$$

□

We now work toward the schematic contex. Recall that X denotes the cross $\mathrm{Spec}(R[st]/(st))$, and R is any ring with reduction being integral.

Lemma 2.6. *Let G be any group. Then* $\mathrm{H}^1_{\mathrm{Zar}}(X, G) = 0$.

Proof. Any Zariski open cover of X can be refined in the following way,

$$U_{\alpha_1} \cup \cdots \cup U_{\alpha_r} \cup V_{\beta_1} \cup \ldots V_{\beta_n} \cup W_{\gamma_1} \cup \cdots \cup W_{\gamma_m}$$

where all opens are connected with non empty connected intersections (for the first type they are all union of two intersecting irreducible components) and

- $U_{\alpha_1} \cup \cdots \cup U_{\alpha_r}$ covers the origin $\mathrm{Spec}(R)$. (Meaning that $\mathrm{Spec}(R) \subset U_{\alpha_1} \cup \cdots \cup U_{\alpha_r}$.)
- $V_{\beta_1} \cup \ldots V_{\beta_n}$ is a cover of the horizontal $\mathbb{G}_{m,R}$ by irreducible opens,
- $W_{\gamma_1} \cup \cdots \cup W_{\gamma_m}$ is a cover of the vertical $\mathbb{G}_{m,R}$ by irreducible opens.

Let (φ_{ij}) be a cocycle. Denote accordingly A, B and C subsets of indices of this cover according to the latter. Note that opens type B and C have empty intersection with one another. Also note that opens of type A all intersect opens of type B and similarly all interesect opens of type C. By Lemma 2.2 we can suppose that $\varphi_{ij} = 1$ for all i, j with $i, j \in A$, $i, j \in B$ and $i, j \in C$. Here we use that $\mathrm{Spec}(R)$ and $\mathbb{G}_{m,R}$ are irreducible. Note that $\varphi_{ab} = \varphi_{a'b'}$ and $\varphi_{ac} = \varphi_{a'c'}$ for all $a, a' \in A$, $b, b' \in B$ and $c, c' \in C$. Indeed

$$\varphi_{a'c'} = \varphi_{a'a}\varphi_{ac}\varphi_{cc'} = \varphi_{ac}$$

and symmetrically for opens of type A and B. So basically we are left with two cocycles φ_{ab} and φ_{ac}. Leting $h_b = \varphi_{ab}$ and $h_c = \varphi_{ac}$ and $h_a = 1$ for any $a \in A$, $b \in B$ and $c \in C$ we see that the initial cocyle is cohomologous to 1 by letting $\varphi'_{ij} = h_i \varphi_{ij} h_j^{-1}$. □

Recall that $C = \mathrm{Proj}(R[xyz]/(xyz))$ denote the triangle. Let $U_\alpha = D_+(\alpha)$ for $\alpha = x, y, z$.

Lemma 2.7. *Let G be any group. Then*

$$\mathrm{H}^1_{\mathrm{Zar}}(C, G) = \mathrm{Conj}(G).$$

Proof. Lemma 2.6 shows that $\mathrm{H}^1(U_x, G) = \mathrm{H}^1(U_y, G) = \mathrm{H}^1(U_z, G) = 0$. Note that $U_x \cap U_y \cap U_z = \emptyset$. Therefore the proof is the same as in Lemma 2.5. □

3. Obstruction to representability

3.1. **Topological spaces.** We do for now some pure topology, proving a bit more than in the general context Proposition 3.3. We will see later what can be adapted to the above case. We work in a cartesian closed category of topological spaces.

Proposition 3.1. *Let $\mathcal{M}$ be a moduli functor on* Top *for the standard topology. If there exists $X \in \mathcal{M}(*)$ such that there exists in $G = \mathcal{M}(*)(X, X) \subset \mathrm{Aut}(X)$ an element which is not homotopic to the identity in $\mathcal{M}(*)(X, X)$ (here $\mathrm{Aut}(X)$ comes equipped with the natural compact open topology and $\mathcal{M}(*)(X, X)$ with the subspace topology) then M is not a sheaf. In particular, it is not representable.*

The proof is similar to the following fact on X-bundles. We recall some about them.

(1) An X-bundle over S is a topological space $p \colon V \to S$ such that there is an open cover U_i of X with $V_i = p^{-1}(U_i) \cong U_i \times X$ as an U_i-topological space.
(2) A morphism of X-bundles over S is simply a morphism over S.
(3) Write $\varphi_i \colon V_i \to U_i \times X$ for the isomorphism. Note that $\varphi_i \varphi_j^{-1} \colon U_{ij} \times X \to U_{ij} \times X$ gives $\varphi_{ij} \in \mathrm{Top}(U_{ij}, \mathrm{Aut}(X))$. These are cocyles with coefficient in the sheaf $\underline{\mathrm{Aut}(X)}$ of continuous maps to the topological group $\mathrm{Aut}(X)$. The following lemma is interesting.

Lemma 3.2. *Two X-bundles over S are isomorphic if and only if they are in the same class in $\mathrm{H}^1(S, \underline{\mathrm{Aut}(X)})$.*

Proof. First, suppose that $\psi \colon V \to W$ is an isomorphism of X-bundles over S. Denote by (φ_{ij}) and (φ'_{ij}) cocycles of V and W respectively. Note that $\varphi'_i \psi \varphi_i^{-1} \colon U_i \times X \to U_i \times X$ gives an element $h_i \in \mathrm{Aut}(X)(U_i)$. We have

$$h_i \varphi_{ij} = \varphi'_i \psi \varphi_i^{-1} \varphi_i \varphi_j^{-1} = \varphi'_i \psi \varphi_j^{-1} = \varphi'_i \varphi_j'^{-1} \varphi'_j \psi \varphi_j^{-1} = \varphi'_{ij} h_j.$$

On the other hand if two X-bundles define the same cohomology class, then there is a covering U_i and $h_i \in \mathrm{Aut}(X)(U_i)$ with $h_i \varphi_{ij} = \varphi'_{ij} h_j$. Now, $\psi_i \colon V_i \to W_i$ defined by $\psi_i = \varphi_i'^{-1} h_i \varphi_i$ glues to a map $\psi \colon V \to W$. Indeed

$$\varphi_i'^{-1} h_i \varphi_i = \varphi_i'^{-1} h_i \varphi_i \varphi_j^{-1} \varphi_j = \varphi_i'^{-1} \varphi'_i \varphi_j'^{-1} h_j \varphi_j = \varphi_j'^{-1} h_j \varphi_j$$

□

(4) Note that in Top any sub-presheaf of a topological space T, say $F \subset h_T$ is representable by the topological space $A := F(*) \subset T$ with the sub-space topology. Indeed a map from a space $Y \to T$ is a a continuous map $Y \to X$ which factors through A. In particular $\mathcal{M}(-)(- \times X, - \times X)$ is the functor representing G.

Proof of proposition 3.1. Let $\sigma \in \mathcal{M}(*)(X, X)$ be an automorphism not homotopic to the identity. Let $E \to S^1$ be the Möbius band construction. We have $E \in \mathcal{M}(S^1)$ by descent. Suppose by contradiction that M is a sheaf. Then E and $X \times S^1$ are isomorphic with $\psi \in \mathcal{M}(S^1)(E, S^1 \times X)$. We now proceed as in Lemma 3.2. Let U_1, U_2, U_3 be a cover of the circle with connected opens with connected double intersection and empty triple intersection. Suppose that (φ_{ij}) are cocycles in $\underline{G}$ coming from trivializations (this is the case with the Möbius consruction and the trivial one). We claim that $h_i = \varphi_i' \psi \varphi_i$ defines an element in $\mathrm{Top}(U_i, G)$. For this we just need to show that it factors to this subspace. But this is true by stability by pullback and the last point in the above remark Therefore, it follows that E and $S^1 \times X$ define the same class in

$$\mathrm{H}^1(S^1, \underline{G}) = [S^1, BG] = \mathrm{Conj}(\pi_1(BG)) = \mathrm{Conj}(\pi_0(\Omega BG)) = \mathrm{Conj}(\pi_0(G))$$

a contradiction. □

3.2. **General context.** What generalizes in the proof above? Let us think in the setup of a category $\mathcal{C}$ equipped with a Grothendieck topology and fiber products. We first analyze X-bundles. The notion makes perfect sense, and cocycles are tautologically valued in the sheaf

$$U \mapsto \mathrm{Aut}_U(U \times X, U \times X)$$

which deserves the notation $\underline{\mathrm{Aut}(X)}$.

Here is the prototypical proposition that we have mind. This is a scheme of proof and specific proofs in contexts consist of showing that hypothesis of this proposition are met.

Proposition 3.3. *Let $(\mathcal{C}, \tau)$ a category with finite limits equipped with a subcanonical Grothendieck topology and $\mathcal{M}$ a moduli functor. Suppose that there exists a 1-cohomological circle C in $\mathcal{C}$. If there exists an $X \in \mathcal{M}(*)$ such that $\mathcal{M}(-)(- \times X, - \times X)$ is representable by a non-trivial discrete group G, then M is not a sheaf, in particular not representable.*

Proof. Take a non-trivial automorphism in G. Perform a Möbius band construction on C. Then the proof of Proposition 3.1 goes. □

Note the following corollary by just taking the contraposition.

Corollary 3.4. *With context as in Proposition 3.3, if M is representable then for any $X \in \mathcal{M}(-)(- \times X, - \times X)$ is either trivial, or not representable by a discrete group.*

Remark. So we see that *discrete automorphisms* are certain (in the sense “sure”) obstructions to representability. If they were non-discrete, it could be the case that they are trivial in term of our cohomological circle. Suppose for example in the topological case that automorphisms groups are all connected and non-discrete. Then from the point of view of the circle, all bundles on this group are trivial. That is to say that if automorphisms are not discrete, we can not be sure that they obstruct representability because they may be “homotopic to the identity” in the sense that they are not detected by a cohomological circle.

Remark. Given a specific context, two things to prove are hidden in Proposition 3.3. Namely the existence of a 1-cohomological circle and that $\mathcal{M}(-)(-\times X, -\times X)$ is representable by a discrete group.

Example 3.5. Proposition 3.3 shows that the moduli of elliptic curves is not representable. Take the elliptic curve $E : y^2 = x^3 + x + 1$. The only automorphism of elliptic curve is given by $y \mapsto -y$ over any base. This shows that $\mathcal{M}(-)(-\times E, -\times E) = \underline{\mathbb{Z}/2\mathbb{Z}}$.

Example 3.6. Let $\mathcal{C} = \mathrm{Set}$ with the discrete topology and we consider $\mathcal{M}(S)$ to be the groupoid of *relative finite sets*, namely sets $T \to S$ such that each fiber is finite. Then M is representable by $\mathbb{N}$. However except for the empty set and the singleton, finite sets have non-trivial automorphisms! Note that for any group G and any set S we have $\mathrm{H}^1(S, G) = 0$. So the presence of a "non-trivial object in first cohomology" goes together with automorphisms being an obstruction to representability.

3.3. **Schemes.** Our goal is now to show the following.

Proposition 3.7. *Let $\mathcal{M}$ be a moduli functor on schemes. Let $k = \overline{k}$ be an algebraically closed field. If there exists $X \in \mathcal{M}(k)$ finite type and separated such that $\mathcal{M}(k)(X, X)$ is a non trivial and finite group, then M is not a Zariski sheaf. In particular, it is not representable by a scheme.*

Proof. The base k is implicit. We show that M is not a sheaf. To do this it suffices that it is not a sheaf on the category finite type separated k-schemes with the Zariski-topology. That's the category $\mathcal{C}$ we are using.

We need to show that $\mathcal{M}(-)(-\times X, -\times X)$ is representable by the finite group G on this category. Note that $S \in \mathcal{C}$ (so separated), and $\psi \in \mathcal{M}(S)(S \times X, S \times X)$ the set of $s \in S$ such that $\psi(s) = g$ for a $g \in G$ is closed in $X \times S$. Indeed this is because the scheme $X \times S$ is separated and so the set $\psi = \mathrm{id} \times g$ is closed for any $g \in G$. Also note that for any closed point $s \in S$, the restriction ψ_s had to be of the form $\mathrm{id} \times g$ for some g. Therefore, because this scheme is Jacobson, we conclude that those subsets form a finite closed decomposition, and therefore a finite open decomposition of $X \times S$. Projection to S, we get an open decomposition of S, showing that $\mathcal{M}(-)(-\times X, -\times X)$ is representable by the finite group G on this category.

Now we can conclude the proof with the help of the prototypical Proposition 3.3. $\square$

References

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