INTRODUCTORY POINTS IN THE THEORY OF PERFECTOID RINGS

LÉO NAVARRO CHAFLOQUE

Abstract. These notes are meant to lay out basic lemmas in the theory of perfectoid rings (Section 2). The definition used (Definition 2.2) is the one of [BMS19]. In particular we prove the tilting equivalence (Proposition 2.20) in this generality. As an application of the theory, we also included a proof of almost purity theorem for perfectoid fields.

We include at the end a section with recollections on (derived) completion (Appendix A).

Contents

1. Functions of the variable p

If p is a function then I am geometer.

We present in this first section elementary properties of the Witt vectors, which are a mixed characteristic analogue of the ring of formal series $R[[t]]$. Namely, it does so that if R is a perfect $\mathbb{F}_p$-algebra, then elements of the Witt vectors $W(R)$ with coefficients in R can be written as,

$$\sum_{n=0}^{\infty}[r_n]p^n$$

where $[r_n]$ will be lifts of $r_n \in R$ and vanishes vis-à-vis a good notion of p-derivation.

Moreover, we would like that to be able to lift the Frobenius to a ring map in the following way

$$\sum_{n=0}^{\infty}[r_n]p^n \mapsto \sum_{n=0}^{\infty}[r_n^p]p^n.$$

1.1. **δ-rings.** To introduce Witt vectors, it is very convenient to introduce first the appropriate notion of p-derivation, which will detect our "constant functions with respect to p" in our power series ring with variable p. References for this section are [BS22, Section 2] and [Bha, Lecture 2].

Meditation. To this end, we will meditate a bit starting from the story of classical derivations, and how from them one gets to the ring of formal series. The reader in a hurry will jump to Definition 1.3.

Let k be a field of characteristic zero. In this brief reminder, we work in the category of k-algebras. A derivation on a k-algebra R is a k-linear map $d : R \to R$, which satisfies the Leibniz rule, for all $r, r' \in R$,

$$d(rr') = rd(r') + r'd(r).$$

This corresponds to the choice of a global section of the k-tangent bundle on $\mathrm{Spec}(R)$. We denote by v_d this corresponding section so that $v_d(x)$ is an infinitesimal direction at x. For every point $R \xrightarrow{-(x)} k$, the map,

$$-(x + \epsilon v_d(x)) : R \to k[\epsilon]$$

that sends $f \mapsto f(x + \epsilon v_d(x)) := f(x) + \epsilon d(f)(x)$ is a morphism of k-algebras. Notice that formally playing around with this yields

$$d(f)(x) = \frac{f(x + \epsilon v_d(x)) - f(x)}{\epsilon}.$$

Derivations $R \to R$ are in one to one correspondence with sections $R \to R[\epsilon]$ of the evaluation at 0. Geometrically this corresponds to retraction from an universal one directional first order thickening $\mathrm{Spec}(R[\epsilon]) \to \mathrm{Spec}(R)$.

If we fix a derivation d on R, then note that we have a very natural ring morphism map $R \to R[[t]]$,

$$f \mapsto \sum_{n=0}^{\infty} d^n(f)\frac{t^n}{n!}$$

that sends a function to it's Taylor series. The fact that this is a ring morphism follows from classical computations with Taylor series. Notice that $R[[t]]$ is on a set theoretic level naturally isomoprhic to $R^{\mathbb{N}}$ *via* coefficients of the series, but the transported ring law is not the pointwise law.

A nice way of thinking of this map is as the unit map of a forgetful $\dashv$ co-free adjunction,

$$k - \mathrm{Alg}_d((R, d), A[[t]]) \cong k - \mathrm{Alg}(R, A)$$

where $k - \mathrm{Alg}_d$ denotes *differential k-algebras*, so k-algebras equipped with a derivation.

To continue the digression, we make a more precise analogy of what will happen. They key is the following key similarity

$$k[[t]] \text{ is similar to } \mathbb{Z}_p.$$

One are functions of an "equicharacteristic formal t-neighbourhood around a k-point", and the other are a functions of an "mixed characteristic formal p-neighbourhood around an $\mathbb{F}_p$-point".

Let us consider first $k[[t]]$-algebras R. We consider a *lift ϕ of the identity on the t-fiber*, for any $f \in R$

$$\phi(f) = f + t\delta(f)$$

Assume first that R is t-torsion free so that $\delta(f)$ is uniquely determined. If you want that ϕ is a ring morphism you are lead to the following axioms for δ, that we will then call a t-derivation.

(1) $\delta(0) = \delta(1) = 0$.
(2) For any $f, g \in R$,
$$\delta(f+g) = \delta(f) + \delta(g).$$
(3) For any $f, g \in R$,
$$\delta(fg) = f\delta(g) + g\delta(f) + t\delta(f)\delta(g).$$

Note that with these axioms, for $k[[t]]$-algebras R such that $tR = 0$, we retrieve the classical notion of derivation of k-algebras so that this notion is really a natural generalization of the notion k-derivations of k-schemes to a notion of "derivation" on schemes that are infinitesimal deformation of k-schemes *i.e.* schemes over $k[[t]]$. What is happening has a geometric flavour behind it. Imagine something over $k[[t]]$, so a deformation of k-scheme around the t-fiber. Then imagine an endomorphism of this scheme that fixes the fiber. What this endomorphism does "around" the fiber should define something "tangent to it", and we see this with the above remark.

Now let's turn to $\mathbb{Z}_p$. Our dream is to realize as a sort $\mathbb{F}_p[[p]]$. The first problem is that $\mathbb{Z}_p$ is not $\mathbb{F}_p$-algebra and so we a priori can not make sense of "multiplying elements of $\mathbb{Z}_p$ by element of $\mathbb{F}_p$". However something suffices for our needs.

Lemma 1.1. *There is a unique multiplicative section*

$$[-] : \mathbb{F}_p \to \mathbb{Z}_p.$$

This section has image exactly the elements $f \in \mathbb{Z}_p$ such that $f^p = f$, i.e *zero and $p-1$ roots of unity. If we realize $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$, then this section can be written as taking any lift $n \in \mathbb{Z}$ of an element in $\mathbb{Z}_p$ and sending to*

$$n \mapsto (n, n^p, n^{p^2}, \cdots).$$

Moreover, any $f \in \mathbb{Z}_p$ has a unique writing as,

$$f = \sum_{n=0}^{\infty} [\alpha_n] p^n$$

for a unique sequence $(\alpha_n) \in \mathbb{F}_p^{\mathbb{N}}$.

Proof. Will follow from Lemma 1.18. See Definition 1.19. Same goes for next Lemma 1.2. □

To really understand where the notion of p-derivation comes from, we will need in fact the following generalization of the last lemma. We denote by q a power of p and $\mathbb{Z}_q = \mathbb{Z}_p[\xi_{q-1}]$ and by $\chi(t)$ the $q-1$ cyloctomic polynomnial.

Lemma 1.2. *There is a unique multiplicative section*

$$[-] : \mathbb{F}_q \to \mathbb{Z}_q.$$

This section has image exactly the elements $f \in \mathbb{Z}_p$ such that $f^q = f$, i.e *zero and the $q-1$ roots of unity. If we realize $\mathbb{Z}_q = \varprojlim \mathbb{Z}[t]/(p^n, \chi(t))$, then this section can be described as follows. Take $\overline{x} \in \mathbb{F}_q$. Take any any lift y_n of $\overline{x}^{1/p^n}$. Then,*

$$\overline{x} \mapsto (y_0, y_1^p, y_2^{p^2}, \cdots)$$

Moreover, any $f \in \mathbb{Z}_q$ has a unique writing as,

$$f = \sum_{n=0}^{\infty} [\alpha_n] p^n$$

for a unique sequence $(\alpha_n) \in \mathbb{F}_q^{\mathbb{N}}$.

Now, these should play the role of the "constants", and for a notion of p-derivation on $\mathbb{Z}_p$-algebras (what we seek) we would ask for $\delta([-]) = 0$. We will now try to apply the same technique as exposed above for t-derivations. We will try to guess what is the right thing to do find a p-derivation on $\mathbb{Z}_q$. If $\delta : \mathbb{Z}_q \to \mathbb{Z}_q$ is the desired p-derivation, in analogy we would like to write

$$\phi(f) = \sigma(f) + p\delta(f)$$

for ϕ a ring morphism and σ some expression that we are still seeking for. Note that the only ring morphisms on $\mathbb{Z}_q$ are lift of the powers of the Frobenius. *In particular note that the only lift of the identity is the identity.* Therefore using the exact same idea as for t-derivation would not work, because it would lead to

$$f = f + p\delta(f)$$

and $\delta = 0$. However we have here the minimal choice to lift the Frobenius instead of the identity,

$$\phi(f) = f^p + p\delta(f).$$

Note that for such a δ which is uniquely determined and non-trivial, we indeed have $\delta(f) = 0$ if and only if $f^q = f$, so this localizes the constant functions with respect to p that we highlighted above. We will see in what follows that such a mantra, *i.e.* p-derivations as reminders of lift of the Frobenius, give rise to a rich theory. To end our meditation, we let ourselves contemplate with the journey in mind

$$d(f)(x) = \frac{f(x + \epsilon v_d(x)) - f(x)}{\epsilon} \qquad \text{and} \qquad \delta = \frac{\phi(f) - f^p}{p}$$

We now pass to formal definitions.

Definition 1.3 (p-derivation, δ-ring)**.** Let A be ring. A *p-derivation on A* is a map of sets $\delta : A \to A$ such that

(1) $\delta(0) = \delta(1) = 0$
(2) For any $x, y \in A$,

$$\delta(xy) = x^p \delta(y) + y^p \delta(x) + p\delta(x)\delta(y)$$

(3) For any $x, y \in A$,

$$\delta(x + y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p}.$$

A *δ-ring* is a ring equipped with a p-derivation. A morphism of δ-rings is a morphism of rings that commutes with p-derivations on both rings. The category of delta rings will be denoted by Ring_δ.

Remark. One sees that for any $n \in \mathbb{Z}$

$$\delta(n) = \frac{n - n^p}{p}$$

This shows that this is the unique δ-ring structure on $\mathbb{Z}$ and that $(\mathbb{Z}, \delta_{\mathbb{Z}})$ is initial in the category of δ-rings.

There is an intimate link between p-derivation and lifts of the Frobenius, as explained earlier. We gathr these connetions in the following Lemma.

Lemma 1.4. *Let A be a ring.*

(1) If $\delta : A \to A$ is a p-derivation then $\phi(f) = f^p + p\delta(f)$ is a lift of the Frobenius modulo p.
(2) If A is p-torsion free, this gives a bijection between δ-structures and lift of the Frobenius.

Contrary to the story of t-derivations talked about earlier, p-derivations is something that gives nothing on algebras that are killed by a power of p.

Lemma 1.5. *Let (A, δ) be a δ-ring such that there exists an n with $p^n A = 0$. Then $A = 0$.*

Proof. By induction on n, using that $n = 0$ is trivial. Note that

$$0 = \delta(p^n) = \frac{p^n - p^{np}}{p} = p^{n-1}(1 - p^{p(n-1)-1}).$$

The right hand side is p^{n-1} times a unit. Induction concludes. □

The following basic remark is also important for later.

Lemma 1.6. *Let (A, δ) be a δ-ring and $f \in A$ with p-torsion. Then $\phi(f) = 0$. Therefore if ϕ is injective A is p-torsion free.*

Proof. Say $px = 0$. We may prove that $x = 0$ by localizing at every prime, so we can suppose that A is p-local. Then

$$0 = \delta(px) = x^p\delta(p) + p^p\delta(x) + p\delta(x)\delta(p) = x^p\delta(p) + \phi(p)\delta(x) = x^p\delta(p) + p\delta(x).$$

But as $\delta(p) = 1 - p^{p-1}$, we get $\phi(x) = p^{p-1}x^p = 0$, because $px = 0$ by assumption. □

Lemma 1.7. *Let A be a δ-ring which is p-adically separated and such that the reduction modulo p is reduced. Let $d \in A$ with $\delta(d)$ being a unit (this is true for any distinguished element if the ring is (p, d)-local). Then A is d-torsion free.*

Proof. Say $dx = 0$. Then

$$0 = \delta(dx) = x^p\delta(d) + d^p\delta(x) + p\delta(x)\delta(d) = x^p\delta(d) + \phi(d)\delta(x).$$

So multiplying by $\phi(x)$ and using that $\delta(d)$ is a unit we get that $0 = x^p\phi(x)$. Reducing modulo p, and using that it is reduced implies that $p \mid x$. So $x = px'$ for some x'. Then $pdx' = 0$. By p-torsion freeness and induction, we conclude that $p^n \mid x$ for every $n \geq 1$ and therefore $x = 0$ by separatedness. □

In the analogy $W(R)$ and $R[[t]]$ we now construct the analogue of $R[\epsilon]$. For any ring R, we endow the product $R\times R$ with the following operations. For $(r_0,r_1),(r_0',r_1')$ we define

$$(r_0,r_1)\hat{+}(r_0',r_1')=(r_0+r_0',r_1+r_1'+\frac{r_0^p+r_0'^p-(r_0+r_0')^p}{p})$$

and

$$(r_0,r_1)\hat{\cdot}(r_0',r_1')=(r_0r_0',r_0^pr_1'+r_0'^pr_1+pr_1r_1')$$

Lemma 1.8. *This defines a functor $W_2:\mathrm{Ring}\to\mathrm{Ring}$ such that for any R, ring morphisms $R\to W_2(R)$ which are sections of the first projection are in one to one correspondence with δ-structures on R.*

Proof. The functoriality to sets equipped with two laws is obvious. We need to prove that this two laws form a ring on $W_2(R)$ for any ring R. If R is p-torsion free, one sees that the map $W_2(R)\to R\times R$

$$(r_0,r_1)\mapsto(r_0,r_0^p+pr_1)$$

is injective and that the image is the subring of $R\times R$ with the pointwise law,

$$\{(f,g)\in R\times R\mid g\equiv f^p\mod p\}$$

and that the laws defined above are exactly the ones that we get by transport of structure.

Now in particular we have proven the claim for free rings, also known as polynomial algebras, and so functoriality and evaluation gives what we want. ☐

1.2. **Witt vectors.** The following can be proven without too much difficulty.

Lemma 1.9. *The forgetful functor $\mathrm{Ring}_\delta\to\mathrm{Ring}$ commutes to all limits and colimits.*

Recall from the earlier meditation that the power series ring could be realized as the co-free part of a forgetful $\dashv$ co-free adjunction from differential rings to rings. The adjoint functor theorem for presentable categories allows us to make the following definition.

Definition 1.10 (Witt vectors)**.** The right adjoint to the forgetful functor $\mathrm{Ring}_\delta\to\mathrm{Ring}$

$$W:\mathrm{Ring}\to\mathrm{Ring}_\delta$$

is called the functor of the *ring of Witt vectors.*

Note that for a δ-ring (A,δ) the unit map

$$A\to W(A)$$

is to be understood as sending functions to it's Taylor series with derivative in the p-direction δ.

Remark. It is straightforward that $\mathbb{Z}\{y\}:=\mathbb{Z}[y_0,y_1,\cdots]$ with $\delta(y_i)=y_{i+1}$ is the free δ-ring on one element [1]. Therefore by (co)-Yoneda lemma we get that $\mathbb{Z}\{y\}$ is a co-δ ring in the category of δ-rings and that for any δ-ring A this co-δ-structure yields an isomorphism (natural in A) of δ-rings

$$\mathrm{Hom}_{\mathrm{Ring}_\delta}(\mathbb{Z}\{y\},A)\cong A.$$

[1] up to isomorphism of δ-rings, which could be many, the only important is that $\delta(y_i)$ is algebraically independent from the other and that $\mathbb{Z}[y_0,y_1,\cdots]$ is generated by $(\delta^n(y_0))_{n\geq 0}$.

One can be more precise: playing a Yoneda game we get that the co-addition $\mathbb{Z}\{y\} \to \mathbb{Z}\{u, v\}$ is given by

$$y_0 \mapsto u_0 + v_0$$

and as a ring map as $(\delta^n(u_0 + v_0))_{n\geq 0}$ and the co-multiplication $\mathbb{Z}\{y\} \to \mathbb{Z}\{u, v\}$ is given by

$$y_0 \mapsto u_0 v_0$$

and as a ring map as $(\delta^n(u_0 v_0))_{n\geq 0}$. Funnily the co-$\delta$ map is given by the δ map.

In view of the last remark we can deduce the following on Witt vectors. Indeed, notice that we have natural isomorphisms of functors (of sets) in R,

$$W(R) \cong \mathrm{Hom}_{\mathrm{Ring}_\delta}(\mathbb{Z}\{y\}, W(R)) \cong \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[y_0, y_1, \cdots], R) \cong R^{\mathbb{N}}$$

Therefore we have a natural bijection of functor of sets,

$$W \to (-)^{\mathbb{N}}$$

between Witt-vectors and the countable product. In other words we get a bijection between the underlying set of the ring $W(R)$ defines abstractly as a functor as above and $R^{\mathbb{N}}$.

Lemma 1.11. *If $pA = 0$ then the Frobenius lift on $W(A)$ from the δ structure is equal to $W(F)$ where $F \colon A \to A$ is the absolute Frobenius.*

Proof. Note that

$$\delta^n(\phi(y_0)) = \phi(\delta^n(y_0)) = \phi(y_n) = y_n^p + py_{n+1}$$

because the Frobenius lift of a δ-ring is always a δ-ring morphism. The claim follows. $\square$

We know introduce *ghost coordinates.* To this end we introduce the category Ring_ψ of rings equipped with a fixed endomorphism. Note that we have a functor $\mathrm{Ring}_\delta \to \mathrm{Ring}_\psi$ sending (A, δ) to (A, ϕ). There is a co-free ring equipped with an endomorphism on a ring, namely the countable product and the shift. Therefore from the universal map $p \colon W(A) \to A$ we get a unique morphism in Ring_ψ

$$W(A) \xrightarrow{\omega_\bullet = (\omega_n)} \prod_{\mathbb{N}} A$$

that we call the *ghost coordinates.* By construction we have that $\omega_n = \omega_0(\phi^n)$ where ω_0 just denotes the projection on the first coordinate. Also if $d \colon \prod_{\mathbb{N}} A \to \prod_{\mathbb{N}} A$ denotes the shift $(a_i) \mapsto (a_{i-1})$ we have

$$\omega_\bullet \phi = d\omega_\bullet.$$

Lemma 1.12 (Witt coordinates.)**.** *There is a change of variable of $\mathbb{Z}\{y\} = \mathbb{Z}[y_0, \ldots, y_n, \ldots]$ with $x_0 = y_0$, $x_1 = y_1$ and $x_n \equiv y_n \mod (y_0, \ldots y_{n-1})\mathbb{Z}[y_0, \ldots, y_{n-1}]$ for $n \geq 2$ such that*

$$\phi^n(x_0) = \sum_{i=0}^{n} p^i x_i^{p^{n-i}} =: \omega_n(x_0, \ldots, x_n)$$

Proof. By induction. Note that the collection of subgroups $I_n = (y_0, \ldots y_{n-1})\mathbb{Z}[y_0, \ldots, y_{n-1}]$ satisfy $\delta(I_n) \subset I_{n+1}$.

Note that $x_n = y_n +$something implies that this defines an automorphism. Say x_n is defined. Then

$$\begin{aligned}\phi^{n+1}(x_0) &= \sum_{i=0}^{n} p^i \phi(x_i)^{p^{n-i}} = \sum_{i=0}^{n} p^i (x_i^p + p\delta(x_i))^{p^{n-i}} \\ &= \sum_{i=0}^{n-2} p^i x_i^{p^{n+1-i}} + p^n(x_n^p + p\delta(x_n)) + p^{n+1} i_{n-1}\end{aligned}$$

where $i_{n-1} \in I_{n-1}$ is some element. So the only choice is

$$x_{n+1} = \delta(x_n) + i_{n-1}.$$

Because $i_n = x_n - y_n \in I_n$ we have

$$\delta(x_n) = \delta(i_n) + \delta(y_n) + j_{n+1} = y_{n+1} + \delta(i_n) + j_{n+1}$$

where $j_{n+1} \in I_{n+1}$ so it concludes.

□

Therefore in Witt coordinates, the ghost components map can be written as

$$(a_n) \mapsto (\omega_n(a_0, \dots, a_n)).$$

Definition 1.13 (Vershiebung.)**.** In *ghost coordinates* we define a natural transformation $V : W \to W$ sending $(a_0, a_1, \dots) \to (0, a_0, a_1, \dots)$ and call it the *vershiebung.*

Remark. We see from the form of the polynomial that the ghost coordinates

$$W(A) \xrightarrow{\omega_\bullet = (\omega_n)} \prod_{\mathbb{N}} A$$

are *injective* when A is p-torsion free and *bijective* when p is invertible. This is useful to prove properties of the Witt vectors by proving them on polynomial rings and the proceed by evaluation. This is the main method to prove items of the following lemma. See [GR18, Section 9.3].

Lemma 1.14 (Properties of the Vershiebung)**.** *Let A be any ring. Let $\underline{a}, \underline{b} \in W(A)$.*

(1) We have $\underline{a}V(\underline{b}) = V(F(\underline{a})\underline{b})$.
(2) The map V is a W-module map $V : F_*W \to W$. In particular $\mathrm{Im}(V)$ is an ideal and is the kernel of the natural map $W(A) \to A$. We denote by $V_n(A) = \mathrm{Im}(V^n)$.
(3) We define (as presheaves of rings) $W_n = W/\mathrm{Im}(V^n)$. We have

$$W = \varprojlim_n W_n.$$

(4) For each $n \geq 1$ we have an exact sequence

$$0 \longrightarrow V_n(A)/V_{n+1}(A) \longrightarrow W_{n+1}(A) \longrightarrow W_n(A) \longrightarrow 0$$

and the first term is isomorphic to A as a $W_{n+1}(A)$ module.
(5) We have that $\cdot p = FV$. If $pA = 0$, then also $VF = FV$ and equals $(a_0, a_1, \dots) \mapsto (0, a_0^p, a_1^p, \dots)$.
(6) If $pA = 0$ and A is semi-perfect then $V_n(A) = p^n W(A)$ and $W(A)$ is p-adically complete.

1.3. Tilt.

Definition 1.15 (Tilt)**.** We denote by $(-)^\flat$ the functor from rings to perfect $\mathbb{F}_p$-algebras,

$$A^\flat = \varprojlim_{(-)\mapsto(-)^p} A/pA$$

that sends A to the inverse limit perfection of the reduction modulo p. Elements of the tilt are then of the form $(\overline{a_0}, \overline{a_1}, \dots) \in (A/pA)^{\mathbb{N}}$ such that for $i \geq 1$, we have $\overline{a_i}^p = \overline{a_{i-1}}$. We denote by $\sharp_n : A^\flat \to A/pA$ the n-th projection.

Remark. Note that whenever $A/pA \neq 0$, then $A^\flat \neq 0$, because for $\alpha \in \mathbb{F}_p$, $(\alpha, \alpha, \dots)$ is always an element of the tilt. Note that on perfect $\mathbb{F}_p$-algebras, the tilt functor is naturally isomorphic to the identity functor.

Geometrically, the tilt look at the p-fiber and preserve the points defined by functions that have all p-th roots and glue together all the others.

The following is useful to think of the tilt as system of compatible p-power roots of R, which is a key aspect of the tilt construction.

Lemma 1.16. *Let R and $\varpi \in R$. Suppose that R is ϖ-complete and that ϖ divides p. Then the natural map*

$$\varprojlim_{(-)\mapsto(-)^p} R \to \varprojlim_{(-)\mapsto(-)^p} R/\varpi R$$

is an isomorphism of topological monoids if we equip R with the ϖ-adic topology.

Proof. We advertise that the proof has a similar taste to the proof of Lemma 1.18. We will define an inverse map.

Fact. *Let R be a ring and I and ideal. Then*

$$x \equiv y \mod I \implies x^p \equiv y^p \mod pI + I^p$$

So as $\varpi \mid p$ by hypothesis, we get that if $r \equiv r' \mod \varpi$ then, $r^{p^n} \equiv r'^{p^n} \mod \varpi^{n+1}$. In particular if $(\overline{r_n})_{n\geq 0} \in \varprojlim_{(-)\mapsto(-)^p} R/\varpi R$. Then for any $m \geq n$, if r_n and r_m are arbitrary lifts of $\overline{r_n}$ and $\overline{r_m}$, then as $r_m^{p^{m-n}} \equiv r_n \mod \varpi$,

$$r^{p^m} - r^{p^n} \in (\varpi^{n+1})$$

In particular, as R is supposed ϖ-complete, we get that

$$(\overline{r_n})_{n\geq 0} \mapsto \lim_{n\to\infty} r_n^{p^n}.$$

is a well defined multiplicative map $\varprojlim_{(-)\mapsto(-)^p} R/\varpi R \to R$. Notice how the above argument does not depend on the choice of representative, so that the limit of two different sequences will different choices of lifts will lie in (ϖ^n) for all $n \geq 0$, and therefore zero. As in $\varprojlim_{(-)\mapsto(-)^p} R/\varpi R$ the p-power is an isomorphism, it suffices by the universal property of $\varprojlim_{(-)\mapsto(-)^p}$ to define a morphism

$$\varprojlim_{(-)\mapsto(-)^p} R/\varpi R \to \varprojlim_{(-)\mapsto(-)^p} R.$$

One easily checks that the map is an inverse to the natural map.

We now show the continuity of the inverse to conclude. It suffices to show the continuity of the above map $\varprojlim_{(-)\mapsto(-)^p} R/\varpi R \to R$. Let $((\overline{r}_{n,k})_{n\geq 0})_k$ be a converging sequence (in k) in $\varprojlim_{(-)\mapsto(-)^p} R$ to $(\overline{r}_n)_{n\geq 0}$. As R is equipped with ϖ-adic topology, $R/\varpi R$ is equipped with the discrete topology and converging sequences are eventually constant sequences. Let N be arbitrary. Let K be enough large such that for all $k' \geq K$ and we have $\overline{r_N} = \overline{r}_{N,k}$. Then we get for all $k' \geq K$

$$\underbrace{(\lim_{n\to\infty} r_{n,k}) - r_{N,k'}}_{\in\varpi^{N+1}} + \underbrace{r_N - \lim_{n\to\infty} r_n}_{\in\varpi^{N+1}} \in \varpi^{N+1}$$

and this implies that the inverse map is continuous. □

Remark. Notice that the proof of the lemma gives furthermore that the following maps are isomorphisms of monoids (and of rings when appropriate)

$$\varprojlim_{(-)\mapsto(-)^p} R \to \varprojlim_{(-)\mapsto(-)^p} R/\varpi pR \to \varprojlim_{(-)\mapsto(-)^p} R/pR \to \varprojlim_{(-)\mapsto(-)^p} R/\varpi R.$$

Note that because $p^2 \subset p\varpi$, it follows that $\bigcap_n (\varpi p) + (p^n) = (\varpi p)$ and therefore that $R/\varpi pR$ is p-complete.[2] So we can apply the above lemma to $\varprojlim_{(-)\mapsto(-)^p} R/\varpi pR \to \varprojlim_{(-)\mapsto(-)^p} R/pR$.

Also the lemma shows that every map is bijective and an appropriate morphism. Note that if $pR = 0$ and that R is ϖ-complete the result always applies.

Corollary 1.17. *Let R be a ϖ-complete ring with $\varpi \mid p$. Then the following addition on $\varprojlim_{(-)\mapsto(-)^p} R$, for $(a_n), (b_n) \in \varprojlim_{(-)\mapsto(-)^p} R$*

$$(a_n) + (b_n) = \left(\lim_{m\to\infty} (a_{m+n} + b_{m+n})^{p^m}\right)$$

gives a ring structure on $\varprojlim_{(-)\mapsto(-)^p} R$ with the pointwise multiplication coming from R. This defines a functor from ϖ-complete rings with $\varpi \mid p$ to perfect rings in characteristic p which is naturally isomorphic to the tilt functor.

There is a nice and fundamental calculation which is an example of the particularly nice interaction of perfect algebras and p-adically complete rings.

Lemma 1.18 (Perfect algebras and p-complete rings)**.** *Let A be a p-adically complete ring. Then there exists a unique multiplicative map $(-)^\sharp : A^\flat \to A$ such that the following diagram commutes,*

$$\begin{tikzcd} A \arrow[dr] & \\ A^\flat \arrow[u, "(-)^\sharp"] \arrow[r, "\sharp_0"'] & A/pA \end{tikzcd}$$

This map can be described in the following way,

$$a^\sharp = \lim_{n\to\infty} \left(\widetilde{\sharp_0(a^{1/p^n})}\right)^{p^n} = \lim_{n\to\infty} \left(\widetilde{\sharp_n(a)}\right)^{p^n}$$

[2]For example because it is p-adically derived complete and p-adically separated, see Lemma A.17.

where $\widetilde{-}$ *denotes taking a lift of an element of* A/pA *in* A*. In words, take the* p^n*-th root of* $a \in A^\flat$ *in* $A^\flat$*, go to* A/pA*, take any lift in* A*, and remultiply by* p^n*. When taking the limit over* n*, this process does not depend on the choice of lift.*

Proof. It is enough to show that there is a unique multiplicative map that fits into the diagram for $n \geq 0$,

$$\begin{tikzcd} A/p^{n+1}A \arrow[dr] & \\ A^\flat \arrow[u, "{[-]_n}"] \arrow[r, "\sharp_0"'] & A/pA \end{tikzcd}$$

Proceed as follows to see the existence. Take any $a \in A^\flat$ and denote by b *any* lift of $\sharp_n(a)$ in $A/p^{n+1}A$. Set $[a]_n = b^{p^n}$. First, note that then the diagram commutes because $\sharp_n^{p^n} = \sharp_0$. To see that is well defined, remark the following – for any ring R and $x, y \in R$, we have,

$$x \equiv y \mod p \implies x^{p^n} \equiv y^{p^n} \mod p^{n+1} \quad \text{(fundamental relation)}$$

Recall again this consequence of the binomial formula.

Fact. *Let* R *be a ring and* I *and ideal. Then*

$$x \equiv y \mod I \implies x^p \equiv y^p \mod pI + I^p$$

If b' was another lift of a $\sharp_n(a)$, we would have $b \equiv b' \mod p$. But then $b^{p^n} \equiv b'^{p^n} \mod p^{n+1}$, which shows that $[-]_n$ is well defined. Now for the uniqueness, if $[-]_n, [-]'_n$ are two such maps, it means that for all $a \in A^\flat$ we have

$$[a]_n \equiv [a]'_n \mod p$$

Using that both maps are multiplicative and the fundamental relation above, one gets that $[a^{p^n}]_n = [a^{p^n}]'_n$. It means that $[-]_n$ and $[-]'_n$ are equalized by precomposition by $(-)^{p^n}$. Using that $(-)^{p^n}$ is an isomorphism on $A^\flat$, we get $[-]_n = [-]'_n$. □

Remark. Let A be a p-adically complete, p-torsion free and residually perfect $\mathbb{Z}_p$-algebra. Then using Lemma 1.18, we see that for any $a \in A$, we have a uniquely determined $b \in A$,

$$a = \overline{a}^\sharp + pb$$

Going by induction, we get that any element can be written uniquely,

$$a = \sum_{n=0}^{\infty} \overline{a_n}^\sharp p^n$$

for elements $(\overline{a_n} \in A/pA)$, with $\overline{a_0} = \overline{a}$.

Definition 1.19 (Teichmüller lift)**.** If $A = W(B)$ for some perfect algebra B in characteristic p, then we denote by $[-]\colon B \to W(B)$ the unique multiplicative section of the projection $W(B) \to B$ obtained by Lemma 1.18.

Theorem 1.20 ([FF18], Proposition 2.1.7)**.** *The Witt vector functor is left adjoint to the tilt functor. For* $A \in \mathbb{Z}_p - \mathrm{Alg}^{\wedge p}$ *and* $B \in \mathrm{Perf}_{\mathbb{F}_p}$ *we have,*

$$\mathbb{Z}_p - \mathrm{Alg}^{\wedge p}(W(B), A) \cong \mathrm{Perf}_{\mathbb{F}_p}(B, A^\flat)$$

The co-unit is given by the following map $\theta_A\colon A_{\mathrm{inf}}(A) := W(A^\flat) \to A$,

$$\sum_{n=0}^{\infty}[a_n]p^n \mapsto \sum_{n=0}^{\infty} a_n^\sharp p^n$$

Proof. We will first define θ_A, and by the doing, show that what is in the statement makes sense: it is a priori not clear that the map is additive (but it is clear that the map is multiplicative). For $n \geq 0$, consider the map $W(A) \to A/p^{n+1}A$,

$$(a_n) \mapsto \omega_n(a_0, \dots, a_n) \mod p^{n+1}$$

where ω_n denotes the n-th ghost component. Note that as $\omega_{n+1}(X_0, \dots, X_{n+1}) = X_0^{p^{n+1}} + p\omega_n(X_1, \dots, X_n)$, one shows by induction on $n \geq 0$ that $\omega_n(pX_0, \dots, pX_n) \equiv 0 \mod p^{n+1}$. Therefore, it follows that we have a well defined map $\psi_n : W(A/pA) \to A/p^{n+1}A$. Consider now the n-th projection map $\sharp_n : A^\flat \to A/pA$. We have $\sharp_n^p = \sharp_{n-1}$ for $n \geq 1$. We look at the composition $\psi_n \circ W(\sharp_n) : W(A^\flat) \to W(A/pA) \to A/p^{n+1}A$. As $\omega_{n+1}(X_0, \dots, X_{n+1}) \equiv \omega_n(X_0^p, \dots, X_n^p) \mod p^{n+1}$, we have that $\psi_{n+1} \circ W(\sharp_{n+1}) \equiv \psi_n \circ W(\sharp_{n+1}^p) = \psi_n \circ W(\sharp_n) \mod p^{n+1}$. Therefore we have a induced map $\theta_A : W(A^\flat) \to A$. We now check that it has indeed the form as in the statement. To show this, it suffices to show that for $a \in A^\flat$ the Teichmüller lift $[a]$ is sent to $a^\sharp$, because p is sent to p. It follows from the fact that $\theta_A \circ [-] \equiv \sharp_0 \mod p$, because of the uniqueness of such multiplicative maps by Lemma 1.18.

We now show that θ_A is indeed a co-unit. To this end, we need to show the following universal property, for any $B \in \mathrm{Perf}_{\mathbb{F}_p}$, $A \in \mathbb{Z}_p - \mathrm{Alg}^{\wedge p}$ and map $\varphi : W(B) \to A^\flat$ there exists a unique map $\overline{\varphi} : B \to A^\flat$ such that,

$$\begin{tikzcd} W(B) \arrow[r, "\varphi"] \arrow[d, "W(\overline{\varphi})"'] & A \\ W(A^\flat) \arrow[ur, "\theta_A"'] & \end{tikzcd}$$

For the existence of $\overline{\varphi}$, note that modulo p, the map $B \to A/pA$ will factor $B \to A^\flat$ by universal limit of the limit perfection. Let this map be $\overline{\varphi}$. Then to show that $\varphi = \theta_A \circ W(\overline{\varphi})$, it suffices to check that $\varphi \circ [-] = \theta_A \circ W(\overline{\varphi}) \circ [-] = \overline{\varphi}^\sharp$. But now for any $b \in B$, we have,

$$\varphi \circ [b] = \varphi(\lim_n (\widetilde{b^{1/p^n}})^{p^n}) = \lim_n (\widetilde{\overline{\varphi}(b)^{1/p^n}})^{p^n} = \overline{\varphi}(b)^\sharp$$

Where we used the (automatic) continuity of φ. The unicity is clear by reducing modulo p.

□

Remark. Note that in the proof we used that a map from Witt vectors of a perfect algebra to a p-adically complete ring is entirely determined by the image of Teichmüller lifts. A universal property can then be reformulated in those terms : for any multiplicative map $B \to A$ such that $B \to A \to A/pA$ is a ring homomorphism, then there exist a unique map $W(B) \to A$ that extends the original multiplicative map via Teichmüller lifts.

The above theorem motivates that we set up some notation.

Definition 1.21 (A_{inf} and Fontaine's θ map)**.** Let R be p-adically complete. We define $A_{\mathrm{inf}}(R) = W(R^\flat)$ and call the counit $\theta\colon A_{\mathrm{inf}}(R) \to R$ *Fontaine's θ map.*

The following corollary gives crucial insight in regard of the interaction between the perfect world in characteristic p and the mixed characteristic $(0, p)$ world.

Corollary 1.22. *The adjunction of Theorem 1.20 restricts to an equivalence of categories between p-adically complete, p-torsion free (*i.e. *flat) and residually perfect $\mathbb{Z}_p$-algebras and perfect $\mathbb{F}_p$-algebras. Moreover, these algebras necessarly carry a unique perfect δ-structure and every ring morphism between such rings will automatically be a δ-ring map.*

Proof. Say R is residually perfect. So $R/pR = R^\flat$. It suffices to show that the co-unit map

$$\theta\colon A_{\mathrm{inf}}(R) \to R$$

is an isomorphism because the unit map is an isomorphism in the case of Theorem 1.20 already. Because $A_{\mathrm{inf}}(R)$ and R are p-torsion free and p-adically complete, it suffices to check that this is an isomorphism modulo p, which is the case as $R/pR = R^\flat$. Indeed by derived Nakayama Lemma A.24 it suffices to check that the map is an isomorphism after $(-) \otimes^L \mathbb{Z}/p$. But by p-torsion freeness this is just the reduction modulo p. The first claim then follows.

Now note that because we just showed that the reduction modulo p for flat, p-complete residually perfect algebras is an equivalence, such rings admit a *unique* lift of the Frobenius who will be an automorphism. Also, every map between those rings are necessary lifts a unique of maps in characteristic p, so they also necessary commute with Frobenii lifts.

□

The following reveals a property about the tilt of a p-complete ring. Namely, it automatically has some completeness property.

Lemma 1.23. *(1) Let B be in characteristic p. Then if $d \in \ker(B^\flat \to B)$, $B^\flat$ is d-complete.*
(2) Let R be p-adically complete and $[a]+px \in \ker(\theta\colon A_{\mathrm{inf}}(R) \to R)$. Then $R^\flat$ is a-complete.

Proof. We first prove (1). Such an element d can be written as $d = (x_i)_{i\geq 1} \in B^\flat$ such that $x_1 = 0$ and $x_i^p = x_{i-1}$ for $i \geq 2$. But then powers of d are of the form

$$(0, 0, 0, 0, \dots, 0, x_2, \dots)$$

implying d-completeness. Indeed, a Cauchy sequence will define by induction a unique element in $B^\flat$.

For the second part, note that $a \in \ker(R^\flat \to R/pR)$.

□

Lemma 1.24. *Let R be ϖ-complete for some $\varpi \mid p$. Suppose that ϖ as (up to a unit) a compatible system of p-power roots $\varpi^\flat \in R^\flat$. Then $R^\flat$ is $\varpi^\flat$-complete.*

Proof. Consequence of Lemma 1.23. Indeed by assumption $\varpi^\flat \in \ker(R^\flat \to R/pR)$. □

Lemma 1.25. *Let R be p-adically complete and $d \in \ker(\theta\colon A_{\mathrm{inf}}(R) \to R)$. Then $A_{\mathrm{inf}}(R)$ is (p, d)-complete.*

Proof. Consequence of item (2) of Lemma 1.23 and Lemma A.10. □

The following yields also a connection between tilt and completions.

Lemma 1.26. *Let B be a perfect algebra in characteristic p. Let $b \in B$ then we have a natural isomorphism*

$$(B/(b))^\flat \to (B/(b))^{\wedge, b}$$

Proof. Look, the morphism of diagrams below is an isomorphism.

$$\begin{tikzcd} {} \arrow[r] & B/(b) \arrow[r, "(-)^p"] \arrow[d, "(-)^{p^3}"] & B/(b) \arrow[r, "(-)^p"] \arrow[d, "(-)^{p^2}"] & B/(b) \arrow[r, "(-)^p"] \arrow[d, "(-)^p"] & B/(b) \arrow[d, "="] \\ {} \arrow[r] & B/(b^{p^3}) \arrow[r] & B/(b^{p^2}) \arrow[r] & B/(b^p) \arrow[r] & B/(b) \end{tikzcd}$$

□

We also note the following property about how the Fontaine's θ map interacts with units.

Lemma 1.27. *Let R be p-adically complete. An element $v \in A_{\inf}(R)$ is invertible if and only if $\theta(v) \in R$ is invertible.*

Proof. We show the non-trivial direction. Write $v = \sum_i [v_i] p^i$. Suppose that the image $\sum_i v_i^\sharp p^i$ is invertible, implying that $v_0^\sharp$ is invertible by p-completeness. But v_0 seen in $\varprojlim_{(-)\mapsto(-)^p} R$ has first component $v_0^\sharp$. As v_0 is a compatible system of p-th power roots of $v_0^\sharp$ this implies that each component is invertible and therefore that v_0 is invertible in $R^\flat$. By p-completeness of $A_{\inf}(R)$, we conclude that v is invertible.

□

2. Perfectoids

(. . .) so in other words, you can think of characteristic p geometry as some infinite covers of characteristic zero geometry over maps which introduces more and more p-power roots of the coordinates. – *Peter Scholze*

The preceding discussion shows hints that the complete mixed characteristic world is connected to the perfect world in characteristic p. We will now continue to explore this kind of reasoning. In the early 80's Fontaine and Wittenberger showed the following theorem.

Theorem 2.1 (Fontaine-Wittenberger, [FW79])**.** *Choose an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. Then there is a canonical isomorphism of Galois groups*

$$\mathrm{Gal}\left(\overline{\mathbb{Q}_p} \mid \mathbb{Q}_p(p^{\frac{1}{p^\infty}})\right) \cong \mathrm{Gal}\left(\overline{\mathbb{F}_p((t))}^{sep} \mid \mathbb{F}_p((t))\right).$$

Geometrically, this suggests more generally an equivalence of an étale topoi in mixed characteristic and an étale topoi in characteristic p. With the goal of proving the weight monodromy conjecture[3], Scholze shed light on the above theorem by generalizing it and finding a geometric[4] setup where to prove it [Sch11]. The first part of this section will be a presentation of the notion of perfectoids rings and will conclude in the Almost purity Theorem 2.36 for perfectoid fields, which is a generalization of Fontaine-Wittenberger 2.1.

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[3]Goal partially achieved in [Sch11, Theorem 9.6], where he introduced and used to this aim the notion of perfectoid space.

[4]Fontaine-Wittenberger is a "point" case.

2.1. **Perfectoid rings.** Here comes the central definition. We will progressively see how such rings benefits from a special interaction with characteristic p. We make our own path following roughly [Bha, Lectures 2-4], [BMS19, Section 3] and [CS23, Section 2].

Definition 2.2 (Perfectoid rings)**.** A *perfectoid ring* is a ring R with an element ϖ such that R is ϖ-complete such that

(1) $\omega^p \mid p$
(2) $\theta\colon A_{\mathrm{inf}}(R) \to R$ is surjective and the kernel is principal.

We will not really care if we equip such rings with the ϖ-adic topology.[5] We will treat being ϖ-complete for some element ϖ has a *property* of the rings in question.

Remark. This definition is the most general and sometimes the best to work with in proofs but not the best for making examples. Namely we will see that for p-torsion free perfectoids, there are more amenable conditions to check. We will give examples at this point.

Example 2.3. We can already give the examples (but note that perfectoids are meant to be a generalization of those) of perfect algebras in characteristic p. The first condition is empty and the second one is satisfied, the map θ being just the reduction modulo p in this case.

Remark. Note that every perfectoid ring R is a $\mathbb{Z}_p$-algebra. Indeed, as $(p) \subset (\varpi)$ and that R is supposed (ϖ)-complete, then it is also p-adically complete by Lemma A.7.[6]

Remark. The ring $\mathbb{Z}_p$ is not a perfectoid ring, the ideal (p) being maximal shuts down the possibility for a ϖ to exist as in the definition. Note that the only p-torsion free perfectoid ring which is reduced modulo p is the zero ring. Indeed as $\varpi^p \mid p$, if it would be reduced modulo p then $p \mid \varpi$ but then $p^p \mid p$ implying that p is a unit, a contradiction to completeness.

We now prove a lemma that allows to reformulate the condition that $\theta\colon A_{\mathrm{inf}}(R) \to R$ is surjective.

Lemma 2.4. *The surjectivity of $\theta\colon A_{\mathrm{inf}}(R) \to R$ is equivalent to the three equivalent assertions.*

(1) Every element of $R/p\varpi R$ is a p-th root. (Note that this ring is not necessarily of characteristic p)
(2) Every element of R/pR is a p-th root.
(3) Every element of $R/\varpi^p R$ is a p-th root.

Proof. We first prove that (2) is equivalent to $\theta\colon A_{\mathrm{inf}}(R) \to R$ being surjective. If so, $R^\flat \to R/pR$ is surjective by reducing modulo p. But this surjectivity exactly says (2). But now the implication is reversible by p-completeness.

Now, ut is clear as $\varpi^p \mid p \mid p\varpi$ that (1) implies (2) implies (3). So we only need to prove that (3) implies (1). As R is ϖ-complete and that we suppose that every element of R/ϖ^p is a p-th root we can write any $r \in R$ as

$$r = \sum_{i=0}^{\infty} r_i^p \varpi^{pi}$$

[5]In the setup of perfectoid spaces, the choice of such a ϖ is part of the data. We note that if we work over some perfectoid base, only one choice is needed.

[6]Be careful however, it is not true in general that the p-adic topology is the same as the ϖ-adic topology, meaning that one can make choices of ϖ where the ϖ topology is coarser than the p-adic topology. This is very clear if $pR = 0$, but there is also mixed characteristic examples.

Now, again by ϖ-adic completeness $\sum_{i=0}^{\infty} r_i\varpi^i$ is a well defined element. But this is now a consequence of the binmonial development that

$$\left(\sum_{i=0}^{\infty} r_i\varpi^i\right)^p - \sum_{i=0}^{\infty} r_i^p\varpi^{pi} \in p\varpi R.$$

□

The parameter ϖ is special with respect to p because up to multiplying by an unit, it admits p^n-power roots for all $n \geq 1$, as explains next lemma. Also, actually the same happens with p.

Lemma 2.5. *Let R be a perfectoid ring and ϖ as in the definition. Then there exist units $u, v \in R^\times$ such that $u\varpi$ and vp have a compatible system of p-power roots.*

Proof. Using Lemma 2.4, every element of $R/p\varpi R$ is a p-th root. Therefore, take

$$(x_n) \in \varprojlim_{(-)\mapsto(-)^p} R/p\varpi R$$

such that $p = x_0 \mod p\varpi$. Using the remark after Lemma 1.16, we know that the natural map $\varprojlim_{(-)\mapsto(-)^p} R \to \varprojlim_{(-)\mapsto(-)^p} R/\varpi pR$ is a bijection. Say $(y_n) \in \varprojlim_{(-)\mapsto(-)^p} R$ is the unique pre-image of (x_n). By construction x_0 has a compatible system of p-power roots and $x_0 = p + p\varpi z = p(1 + \varpi z)$, which concludes by completeness. Now note that one could make the exact same argument with ϖ instead of p. □

Corollary 2.6. *In the Definition 2.2, asking (1) is equivalent to ask*

(1') The element ϖ admits a compatible system of p-power roots and $\varpi^p \mid p$
(1") There is some unit u such that $\varpi^p = pu$.
(1"') The element ϖ admits a compatible system of p-power roots and $\varpi^p = pu$ for some unit u.

Also, we can ask that R is p-complete *instead of ϖ-complete for some ϖ that satisfies (1).*

Remark. Note that for different ϖ_1 and ϖ_2 as above, the ϖ_1 and ϖ_2 topology can change. So in context of Tate perfectoid rings the corollary above can be uneasy.

We will now take a closer look to the kernel of θ.

Definition 2.7 (Distinguished element)**.** Let A be a p-local δ-ring. An element $a \in A$ is *distinguished* if $\delta(a)$ is a unit.

Example 2.8. For example, if B is a perfect algebra in characteristic p, then an element in $(b_n) \in W(B)$ is distinguished if and only $b_1 \in B^\times$.

Lemma 2.9. *Let R be perfectoid. Then* $\ker(\theta)$ *is generated by a distinguished element and any distinguished element in the kernel generates it. More precisely if ϖ is as Definition 2.2 so that $\varpi^p(-x) = p$ for some x and admits a compatible system of p-th roots $\varpi^\flat \in R^\flat$, then*

$$\ker(\theta) = (p + [\varpi^\flat]^p y)$$

for any $y \in A_{\inf}(R)$ pre-image of x.

Proof. Let us observe the following. Let $d \in \ker(\theta)$ distinguished. Suppose that there is some $d' \in \ker(\theta)$ and $x \in A_{\text{inf}}(R)$ such that $d = d'x$ in $A_{\text{inf}}(R)$. Then using the law in W_2

$$d_1 - d'_0 x_1 = x_0^p d'_1$$

Because $R^\flat$ is d'_0-complete by Lemma 1.23 we see that x is a unit and that d' is distinguished.

Therefore, to show the claim it suffices to show that there exists a distinguished element in the kernel. Indeed, $\ker(\theta) = (d')$ for some d' because we suppose that this kernel is principal.

To this end we now prove the "more precisely" part of the assertion. The element $p + [\varpi^\flat]^p y$ is clearly in the kernel. In W_2 this reads

$$(0,1) + (\varpi^{\flat p}, 0)(y_0, y_1) = (0,1) + (\varpi^{\flat p} y_0, \varpi^{\flat p^2} y_1) = (\varpi^{\flat p} y_0, 1 + \varpi^{\flat p^2} y_1).$$

Because R is ϖ-complete, $R^\flat$ is $\varpi^\flat$-complete. Indeed, a Cauchy sequence for the $\varpi^\flat$-topology in $R^\flat$ will be point-wise Cauchy for the (ϖ)-topology, which shows our claim. Therefore we can deduce that $p + [\varpi^\flat]^p y$ is indeed distinguished. The first part of the Lemma now concludes. □

Corollary 2.10. *A characteristic p perfectoid is just a perfect algebra.*

Proof. If R is perfectoid and $pR = 0$ then $p \in \ker(\theta)$ implying by Lemma 2.9 that $(p) = \ker(\theta)$, concluding. □

The following is a key aspect of perfectoids rings.

Proposition 2.11. *Let R be perfectoid and ϖ as in Definition 2.2. Then*

$$(-)^p \colon R/\varpi R \to R/\varpi^p R$$

is an isomorphism.

Moreover if ϖ has a compatible system of p-power roots $\varpi^\flat \in R^\flat$, then we have a square of isomorphisms

$$\begin{array}{ccc} R^\flat/\varpi^\flat R^\flat & \xrightarrow{\sharp} & R/\varpi R \\ \downarrow{\scriptstyle (-)^p} & & \downarrow{\scriptstyle (-)^p} \\ R^\flat/\varpi^{\flat p} R^\flat & \xrightarrow{\sharp} & R/\varpi^p R \end{array}$$

Proof. Because the first statement is insensible to multiplying by units, we can show the second statement instead by supposing that ϖ has a compatible system of p-power roots by Lemma 2.5. Note that in the square, the left vertical map is an isomorphism because $R^\flat$ is perfect.

By Lemma 2.9, we can infer that $\ker(\theta) = (p + [\varpi^\flat]^p y)$ with the same notations. Now consider

$$A_{\text{inf}}(R)/[\varpi^\flat] \to R/\varpi R.$$

By passing to the quotient, we get the desired isomorphism

$$A_{\text{inf}}(R)/(p + [\varpi^\flat]^p y, [\varpi^\flat]) = A_{\text{inf}}(R)/(p, [\varpi^\flat]) = R^\flat/\varpi^\flat R^\flat \to R/\varpi R.$$

Note that the above argument works exactly in the same manner with ϖ^p. Therefore both horizontal maps are isomorphisms, concluding that the right vertical map is also an isomorphism. □

Remark. This lemmas show that a perfectoid ring R and it's tilt $R^\flat$ are pro-infinitesimal deformation of the same ring

$$R^\flat/\varpi^\flat R^\flat \xrightarrow{\sim} R/\varpi R.$$

This hints at the strategy of the proof of almost purity Theorem 2.36.

Remark. Note that the lemma shows that if R is perfectoid and ϖ' is any element such that R is ϖ'-complete and $\varpi'^p \mid p$ then the conclusion of Proposition 2.11 holds. For example if the conclusion holds for ϖ, the conclusion holds for ϖ^{1/p^n} if it exists, giving a sequence of isomorphisms of p-power maps

$$R/\varpi^{1/p^n}R \xrightarrow{(-)^p} R/\varpi^{1/p^{n-1}} \to \cdots \to R/\varpi R \xrightarrow{(-)^p} R/\varpi^p R.$$

Note also the following observation, which complements the preceding remark.

Lemma 2.12. *Let R be perfectoid such that $pR \neq 0$. Say ϖ is as in Definition 2.2. Then there is an integer k such that ϖ^{p^k} does not divide p. In particular up to changing ϖ, we can always suppose that ϖ^{p^2} does not divide p if $pR \neq 0$.*

Proof. By Lemma 2.5, we can suppose that ϖ has a compatible system of p-power roots. If ϖ^{p^k} always divides p, then we have an isomorphism by Proposition 2.11

$$R^\flat/\varpi^{\flat p^k}R^\flat \xrightarrow{\sharp} R/\varpi^{p^k}R$$

for every integer k. Taking the inverse limit implies that $R^\flat \cong R$ a contradiction to $pR \neq 0$. $\square$

We now give a more convenient criterion to check if a ring is perfectoid when R is ϖ-torsion free. For example, in the p-torsion free case.

Definition 2.13 (p-integrally closed)**.** We say that an inclusion of rings $A \subseteq B$ is *p-integrally closed* if for all $b \in B$ such that $b^p \in A$ then $b \in A$. Note that every integrally closed extension is p-integrally closed.

Lemma 2.14. *Let R be a ring and $\varpi \in R$ a non-zero divisor. Then the p-power map $R/\varpi R \to R/\varpi^p R$ is injective if and only if $R \to R[\frac{1}{\varpi}]$ is p-closed.*

Proof. We begin by supposing that the p-power $R/\varpi R \to R/\varpi^p R$ is injective. Let $x \in R[\frac{1}{\varpi}]$ such that $x^p \in R$. Let $n \geq 0$ be minimal such that $\varpi^n x \in R$. Suppose by contradiction that $n > 0$. We know that $(\varpi^n x)^p \in R$. Therefore by assumption $\varpi^n x \in \varpi R$. As ϖ is a non zero-divisor this contradicts the minimality of n.

We now suppose that $R \to R[\frac{1}{\varpi}]$ is p-closed. Then if $x \in R$ such that $x^p = \varpi^p y$, then $(x/\varpi)^p \in R$, and therefore $(x\varpi) \in R$. $\square$

Proposition 2.15 (Equivalent definition of perfectoids, torsion free case)**.** *Let R be a ϖ-complete ring, where ϖ is a non-zero divisor and $\varpi^p \mid p$. Then*

$$R \text{ is perfectoid} \iff (-)^p\colon R/\varpi R \to R/\varpi^p R \text{ is an isomorphism.}$$

Or, also equivalently, R/pR is semi-perfect and $R \subset R[\frac{1}{\varpi}]$ is p-integrally closed.

Proof. If R is perfectoid, then the second condition is satisfied by Proposition 2.11.

Suppose now that $(-)^p\colon R/\varpi R \to R/\varpi^p R$ is an isomorphism. By Lemma 2.4, wee see that $\theta\colon A_{\mathrm{inf}}(R) \to R$ is surjective. We therefore only need to show that $\ker(\theta)$ is principal.

Note that the proof of Lemma 2.5 applies so that up to multiplying by a unit we can suppose that ϖ has a compatible system of p-power roots $\varpi^\flat$. That $R \subset R[\frac{1}{\varpi}] = R[\frac{1}{\varpi^{1/p^n}}]$ is p-integrally closed implies that $R/\varpi^{1/p^n}R \xrightarrow{(-)^p} R/\varpi^{1/p^{n-1}}$ is injective by Lemma 2.14. Also the surjectivity is implied by the semi-perfectness of R/pR by Lemma 2.4, which is implied by $(-)^p\colon R/\varpi R \to R/\varpi^p R$ being surjective again by Lemma 2.4.

Therefore we conclude that p-power maps

$$R/\varpi^{1/p^n}R \xrightarrow{(-)^p} R/\varpi^{1/p^{n-1}} \to \cdots \to R/\varpi R \xrightarrow{(-)^p} R/\varpi^p R$$

are all isomorphisms. This implies that

$$\sharp\colon R^\flat/\varpi^\flat R^\flat \to R/\varpi R$$

is an isomorphism. Now find, $p + [\varpi^\flat]^p y$ in $\ker(\theta)$ as in Lemma 2.9 with same notations. To show that this element is a generator of the kernel, as $W(R^\flat)$ is $[\varpi^\flat]$-complete by Lemma A.10 and $[\varpi^\flat]$-torsion free because R is ϖ-torsion free, and that R is ϖ-complete, proving the desired isomorphism $W(R^\flat)/(p + [\varpi^\flat]^p y) \to R$ modulo $[\varpi^\flat]$

$$W(R^\flat)/(p, [\varpi^\flat]) \to R/\varpi R$$

is sufficient. Indeed the kernel K of this map is derived complete, and therefore classically complete by Lemma A.17 with the torsion freeness assumption. But also $[\varpi^\flat]K = K$ which implies by separatedness that $K = 0$. Surjectivity will follow from Lemma A.4.[7]

But modulo $[\varpi^\flat]$ this map is exactly the isomorphism $R^\flat/\varpi^\flat R^\flat \to R/\varpi R$ proved above.

□

Example 2.16. We now give examples of perfectoid rings, using Propostion 2.15 as criterion.

(1) *The p-adic completion $\mathbb{Z}_p[p^{1/p^\infty}]^{\wedge,p}$.* Indeed, first this ring is p-complete by construction. We let $\varpi = p^{1/p}$. Let us also precise what we mean by taking p-power roots of p. We realize this has follows[8]

$$\mathbb{Z}_p[p^{1/p^\infty}] = \mathbb{Z}_p[t^{1/p^\infty}]/(t-p).$$

Therefore when quotienting the p-adic completion by p we get

$$\mathbb{F}_p[t^{1/p^\infty}]/(t).$$

Because $\mathbb{F}_p[t^{1/p^\infty}]$ is perfect the Frobenius

$$\mathbb{F}_p[t^{1/p^\infty}]/(t^{1/p}) \to \mathbb{F}_p[t^{1/p^\infty}]/(t)$$

is an isomorphism which concludes by Proposition 2.15.

The tilt of this perfectoid is therefore $\mathbb{F}_p[t^{1/p^\infty}]^{\wedge,t}$ by Lemma 1.26.

(2) *A perfectoid associated to any p-ramified finite extension of $\mathbb{Q}_p$, $\mathcal{O}_K[\varpi^{1/p^\infty}]^{\wedge,p}$.* We now give a generalization of the previous example. Let K be a finite extension of $\mathbb{Q}_p$ with residue field k and integral elements $\mathcal{O}_K$ and ϖ an uniformizer in $\mathcal{O}_K$. Suppose that the degree of ramification of K is at least p, meaning that $\varpi^p \mid p$. Then we consider

$$\mathcal{O}_K[t^{1/p^\infty}]/(t-\varpi)$$

[7]One could also say that because for $[\varpi^\flat]$-torsion free modules M, then $M \otimes W(R^\flat)/(\varpi^\flat) = M \otimes^L W(R^\flat)/(\varpi^\flat)$, we can use derived Nakayama Lemma A.24 to conclude.

[8]where $\mathbb{Z}_p[t^{1/p^\infty}]$ is the colimit of $\mathbb{Z}_p[t]$ under $t \mapsto t^p$.

that we denote by $\mathcal{O}_K[\varpi^{1/p^\infty}]$. We claim that $\mathcal{O}_K[\varpi^{1/p^\infty}]^{\wedge,p}$ is perfectoid. Quotienting by ϖ before the p-adic ($=\varpi$-completion) gives

$$k[t^{1/p^\infty}]/(t).$$

Therefore, again, because $k[t^{1/p^\infty}]$ is perfect, the map

$$(-)^p \colon k[t^{1/p^\infty}]/(t^{1/p}) \to k[t^{1/p^\infty}]/(t)$$

is an isomorphism, proving our claim by Proposition 2.11. Note that here we see that it is useful to take some ϖ such that the p-th power is not p times a unit to prove our various above lemmas. Note that this association is functorial in ramified extensions of $\mathbb{Q}_p$ of ramification degree at least p. Similarly as above, by Lemma 1.26, the tilt of this perfectoid is $k[t^{1/p^\infty}]^{\wedge,t}$.

(3) *Integral elements of* $\mathbb{C}_p$, $\mathcal{O}_{\mathbb{C}_p}$. Let $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$ the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. This is again algebraically closed, see Proposition 2.25. This is an algebraically closed field of the same cardinlity as $\mathbb{C}$ so is actually isomorphic to $\mathbb{C}$ as field. Consider $\mathcal{O}_{\mathbb{C}_p}$ the ring of elements of p-adic valuation less or equal to 1. This is a local ring, and the maximal ideal is given by (p^{1/p^∞}). The residue field is $\overline{\mathbb{F}}_p$. Because every element has a p-power root because it is algebraic closed, we see that $\mathcal{O}_{\mathbb{C}_p}$ is semi-perfect. Also, it is integrally closed in $\mathbb{C}_p$ implying that it is p-integrally closed, and therefore perfectoid.

Now to determine it's tilt, we actually refer to the proof of Theorem 2.36. Namely as $\mathbb{C}_p$ is the completion of the algebraic closure of $\mathbb{Q}_p(p^{1/p^\infty})^{\wedge,p}$, the tilt of $\mathbb{C}_p$ is the completion of the algebraic closure of $\mathbb{F}_p(t^{1/p^\infty})$. So

$$\mathbb{C}_p^\flat = \widehat{\overline{\mathbb{F}_p(t)}}$$

where the completion is t-adic. Taking $\mathcal{O}_{\widehat{\overline{\mathbb{F}_p(t)}}}$ gives the tilt of $\mathcal{O}_{\mathbb{C}_p}$.

(4) Take k to be a perfect field. Consider the mixed characteristic local ring

$$A = W(k)[[x_2, \ldots, x_d]]$$

with parameters $x_1 = p, x_2 \ldots, x_d$. We claim the the p-adic completion B of

$$B' = A[p^{1/p^\infty}, x_2^{1/p^\infty}, \ldots, x_d^{1/p^\infty}]$$

is perfectoid. That it is residually perfect is clear. We take $\varpi = p^{1/p}$. We show that B' ring is p-integrally closed. This suffices because we care about the injectivity of $(-)^p \colon B/\varpi B = B'/\varpi B' \to B'/\varpi^p B' = B/\varpi B$. Note first that as $W(k) \subset W(k)[\frac{1}{p}]$ is integrally closed, if a constant is in $B'[\frac{1}{p}]$ then it is actually in B'. Note that any element in this ring which is not constant has a *minimal positive degree.* The minimal degree (in x_2 say) coefficient of the p-th power of an element will be the p-th power of the coefficient of the minimal degree. Therefore using the constant case we get that this coefficient is already in B. This shows by induction that B is p-integrally closed and therefore perfectoid.

Using the same technique as above, we see that

$$B^\flat = k[[t, x_2, \ldots, x_d]][t^{1/p^\infty}, x_2^{1/p^\infty}, \ldots, x_d^{1/p^\infty}]^{\wedge,t}$$

(5) *The p-adic completion $R[t^{1/p^\infty}]^{\wedge,p}$, for a p-torsion free perfectoid ring R.* Indeed, this ring is p-adically complete and semi-perfect modulo p. Say $\varpi^p = pu$ and ϖ admits a compatible sequence of p-power roots. This is sometimes call the *perfectoid* $\mathbb{A}^1$ although we stress that this not the initial perfectoid over $R[t]$. (One can proof using prismatic cohomology to see that there is no such ring.) The same argument with the minimal degree concludes that this is perfectoid (using this times that R is integrally p-closed).

Note that $R^\flat[t^{1/p^\infty}]$ is perfect and then by Lemma 1.23 $(R^\flat/\varpi^\flat[t^{1/p^\infty}])^\flat = R^\flat[t^{1/p^\infty}]^{\wedge,\varpi^\flat}$.

The following is good to have in mind.

Lemma 2.17. *There is no initial perfectoid ring.*

Proof. First note that $\mathbb{F}_p$ is perfectoid ring. So if an initial perfectoid ring R would exist then there is a map $R \to \mathbb{F}_p$, necessarily surjective of kernel p. But then $A_{\inf}(R) = \mathbb{Z}_p$. Because of the existence of the map $R \to \mathbb{F}_p$, $\ker(\theta) \subset (p)$. But because $\ker(\theta)$ is principal and generated by a distinguished element by Lemma 2.9, we conclude that $\ker(\theta) = (p)$. But we have seen in Example 2.16 that there are p-torsion free perfectoids.

□

We now address the fact that the ϖ-torsion in perfectoid rings is really tame.

Lemma 2.18 (Bounded torsion)**.** *Let B be a perfect algebra in characteristic p which b-complete for some $b \in B$. Then*

(1) $d = [b]x + p \in W(R)$ is distinguished for any $x \in W(R)$,

(2) the ring $R = W(B)/d$ has bounded p^∞-torsion. More precisely

$$R[p] = R[p^\infty].$$

(3) The ring R is p-adically complete.

(4) Denote by ϖ the image of $[b]$ in R. Then R has bounded ϖ-torsion. More precisely

$$R[\varpi^{1/p^\infty}] = R[\varpi] = R[\varpi^\infty].$$

(5) The ring R is ϖ-adically complete.

Proof. For item (1), proceed in the same manner as in the proof of Proposition 2.11.

For item (2), we show that if $p^2x = dy$ in $W(B)$, then $p \mid y$, implying by p-torsion freeness that $px = dy'$ for some y' and implying that $R[p] = R[p^\infty]$. Applying δ, we get that $\delta(dy) \in pW(B)$. But

$$\delta(dy) = y^p\delta(d) + \delta(y)\phi(d).$$

Multiplying by $\phi(y)$ and using that $\delta(d)$ is a unit, we get that $\phi(y)y^p = 0$ in B, giving the claim.

Now as $W(B)$ is p-complete, R is derived p-complete by Lemma A.21. But by (2) and Lemma A.17, we deduce that R is classically p-complete.

Because $W(B)$ is p and d-torsion free by Lemmas 1.6 and 1.7 we get by torsion exchange

$$R[p] = W(B)/(d)[p] \cong W(R)/(p)[d] = B[d]$$

where the isomorphism in the middle is in isomorphism of $W(B)$-modules. Now note that as $\varpi \mid p$ we have $R[\varpi^n] \subset R[p^n] = R[p]$. By the isomorphism above $R[\varpi^n]$ is sent to $B[b^n]$, but now Lemma A.18, concludes.

The proof item (5) is now the same the proof for item (3).

□

Let us collect results from Lemma 2.18 for perfectoid rings in the following Lemma.

Lemma 2.19. *Let R be a perfectoid ring and ϖ as in Definition 2.2. Then*

(1) We have $\sqrt{\varpi R} = (\varpi^{1/p^\infty})$. In particular $\sqrt{\varpi R}^2 = \sqrt{\varpi R}$.

(2) All the inclusions

$$R[\sqrt{\varpi R}] = R[\varpi^{1/p^\infty}] \subset R[\varpi] \subset R[\varpi^\infty]$$

are equalities.

(3) If ϖ^p is p times a unit, we can say that all the inclusions

$$R[\sqrt{pR}] = R[\varpi^{1/p^\infty}] \subset R[p] \subset R[p^\infty]$$

are equalities.

In particular any perfectoid as bounded p^∞-torsion and bounded ϖ^∞-torsion.

Proof. To prove the first item, it suffices to show that $R/(\varpi^{1/p^\infty})$ is reduced. But this is $A_{\inf}(R)/(d, [a_0^{1/p^\infty}])$ if $d = [a_0]x + p$ for some element x. But then $R/(\varpi^{1/p^\infty}) = R^\flat/(a_0^{1/p^\infty})$ which is perfect, therefore reduced.

The second and third claim are direct consequences of Lemma 2.18. Indeed, the assertions are insensible by unit multiplication so we can use Lemma 2.5 and d constructed as Lemma 2.9 to apply Lemma 2.18.

$\square$

The following is a first step in Theorem 2.36, the almost purity theorem.

Proposition 2.20 (Tilting equivalence)**.** *Let R be perfectoid and ϖ as in Definition 2.2 admitting a compatible system of p-power roots $\varpi^\flat \in R^\flat$. Let $d \in A_{\inf}(R)$ a generator of* $\ker(\theta\colon A_{\inf}(R) \to R)$*. Then, we have the following equivalences of categories, with opposing arrows denoting mutually inverse functors.*

$$\begin{tikzcd}
\{R \to S\ \varpi\text{-complete perfectoid } R\text{-algebras}\} \arrow[d, shift right, "A_{\inf}"'] \arrow[dd, bend right=60, "\flat"'] \\
\{A_{\inf}(R) \to D\ (p, [\varpi^\flat])\text{-complete perfect } \delta\text{-}A_{\inf}(R)\text{-algebras}\} \arrow[u, shift right, "\bmod d"'] \arrow[d, shift right, "\bmod p"'] \\
\{R^\flat \to S'\ \varpi^\flat\text{-complete perfectoid } R^\flat\text{-algebras}\} \arrow[u, shift right, "A_{\inf}"'] \arrow[uu, bend right=60, "\sharp"']
\end{tikzcd}$$

The left composite $\sharp$ is called the untilt*.*

Proof. Assuming that functors are well defined, we show that they are mutual inverse of categories. It suffices to treat the case $A_{\inf}$ and modulo d, because $A_{\inf}$ and modulo p is a special case. Note that θ gives a natural transformation in $R \to S$ perfectoid $A_{\inf}(S)/d \to S$. But because the image of a distinguished element by a δ-map is distinguished, we conclude that this map is an isomorphism by Lemma 2.9.

The natural transformation θ also gives what we want on the other side. Namely

$$\theta\colon A_{\inf}((D/d)^\flat) \to D$$

is an isomorphism using the identification $(D/d)^\flat = D/p$ by Lemma 1.26 and Corollary 1.22.

We now show that functors are well defined. Take some ϖ-complete R-algebra S. Then by Lemmas and 1.24 and A.10, we see that $A_{\text{inf}}(S)$ is $(p, [\varpi^\flat])$-complete. Therefore the A_{inf} functor is well defined.

We show that the functor modulo d is well defined. Let D a $(p, [\varpi^\flat])$-complete perfect δ-$A_{\text{inf}}(R)$-algebra. First, by Corollary 1.22 we can replace D by $W(D/p)$. Note that D/p is derived $\varpi^\flat$-complete by Lemma A.21, but then classically $\varpi^\flat$-complete by Lemma A.18. Therefore by Lemma 2.18 we deduce that $D/d = W(D/p)/d$ is ϖ-complete. To show that D/d is perfectoid we need to show that the kernel of Fontaine's θ map is principal. But chasing the Teichmüller lifts, we deduce that the following diagram commutes

$$\begin{tikzcd}
W((D/d)^\flat) \arrow[d, "\sim"'] \arrow[dr, "\theta"] & \\
W(D/p) \arrow[d, "\sim"'] & D/d \\
D \arrow[ur, "\mathrm{mod}\ d"'] &
\end{tikzcd}$$

giving the claim.

□

Remark. We can not avoid any completeness assumption. If ϖ is such that $\varpi^p = pu$ for some unit u, we can rewrite the equivalence as

$$\begin{tikzcd}
\{R \to S \text{ perfectoid } R\text{-algebras}\} \arrow[d, shift right, "\flat"'] \\
\{R^\flat \to S' \ \varpi^\flat\text{-complete perfectoid } R^\flat\text{-algebras}\} \arrow[u, shift right, "\sharp"']
\end{tikzcd}$$

So we see that that the natural topology on the tilt is an important data. If there is no topology mentioned in the first category mentionned in this remark, it is just that the p-complete topology is implicitly used and not mentionned.

Tilting equivalence 2.20 implies a new equivalent definition of perfectoid rings.

Corollary 2.21 (Perfectoid rings are Hodge-Tate loci of perfect prisms)**.** *A perfectoid ring is a ring of the form $W(B)/d$ where B is perfect and d is distinguished with the property that B is d-complete. This means, if write $d = [b] + px$, that B is b-complete.*

2.2. **Perfectoid fields.** Before defining perfectoid fields, we do some recollection on complete valued fields.

2.2.1. *Results on complete valued fields.*

Lemma 2.22. *Let F be a complete valued field. Let L be an algebraic extension. Then there exists a unique extension of the valuation of K to L. Namely if $x \in F'$ where F' is a finite extension if $n = [F' : F]$ then $|x| = |N_{F'|F}(x)|_F^{\frac{1}{n}}$. In particular the norm is invariant by F-automoprhisms.*

Lemma 2.23 (Krassner's lemma)**.** *Let F be a complete valued field. Let $\alpha, \beta \in F^{sep}$. If*

$$|\alpha - \beta| < |\alpha - \alpha'|$$

for every distinct conjugates α' of α, then $\alpha \in F(\beta)$.

Proof. Let H be the Galois subgroup of the absolute galois group that correspond to the extension $F(\beta)$. We want to show that α is fixed by every $\sigma \in H$. By contradiction, suppose that $\sigma(\alpha) \neq \alpha$. Then we have

$$|\alpha - \sigma(\alpha)| = |\alpha - \beta + \beta - \sigma(\alpha)| = |\alpha - \beta - \sigma(\alpha - \beta)| \leq \max\{|\alpha - \beta|, |\sigma(\alpha - \beta)|\}.$$

By invariance by automorphisms we have

$$|\alpha - \sigma(\alpha)| \leq |\alpha - \beta| < |\alpha - \sigma(\alpha)|,$$

a contradiction. □

Corollary 2.24 (Continuity of separable extensions)**.** *Let F be a complete valued field. Let $f(t) \in F[t]$ be an irreducible separable monic polynomnial, with roots (α_i). Then for every $\epsilon > 0$ sufficiently small, there exists a $\delta > 0$ such that for every monic polynomial $g \in F[t]$ of the same degree as f with $\|f - g\| < \delta$ then there is a numbering of the roots (β_i) of g such that $|\alpha_i - \beta_i| < \epsilon$ and $F(\alpha_i) = F(\beta_i)$. Also, g is irreducible and separable.*

Proof. As the roots vary continuously in the coefficients, we can arrange that

$$|\alpha_i - \beta_i| < \min_{i \neq j}(|\alpha_i - \alpha_j|).$$

Then by Krassner's lemma, we see $F(\alpha_i) \subset F(\beta_i)$. But as

$$[F(\beta_i) : F] \leq \deg(g) = \deg(f) = [F(\alpha_i) : F],$$

we deduce $F(\alpha_i) = F(\beta_i)$. □

Proposition 2.25. *Let F be a complete valued field and $F_0 \subset F$ a dense subfield. Then F is separably closed if and only if F_0 is.*

2.2.2. *Perfectoid fields.*

Remark. Note that a perfectoid ring which is a field is automatically discrete in characteristic p. Therefore we see that Proposition 2.20 is basically not interesting in this case. However as seen in Example 2.16, interesting fields appear when we invert ϖ (or p) on a perfectoid local ring. So see next definition and beware, a perfectoid field is not a field which is a perfectoid ring.

Definition 2.26 (Perfectoid field)**.** A *perfectoid field* is a complete non-archimedan field K with residue field of characteristic p and rank 1 valuation such that bounded elements K° is a perfectoid ring. (This implies that the associated rank 1 valuation is non discrete).

Remark. In this setup, we always have $K^\circ[\frac{1}{\varpi}] = K$ for any ϖ such that $|\varpi| < 1$.

Note that an immediate application of the tilting equivalence 2.20 for perfectoid rings yields (inverting ϖ and $\varpi^\flat$)

Proposition 2.27 (Tilting equivalence – field extensions)**.** *Let K be a perfectoid field. Then the tilt functor is an equivalence of categories,*

$$\{\textit{Perfectoid fields over } K\} \to \{\textit{Perfectoid fields over } K^\flat\}.$$

Next lemma follows for example from properties of the untilt map on the level of ring of integers.

Lemma 2.28. *Let K be a perfectoid field and $K^\flat$ it's tilt. If $|\cdot|$ is a rank 1 valuation on K that gives K it's topology, then $|\cdot| \circ \sharp$ is a rank 1 valuation on $K^\flat$ that gives $K^\flat$ it's topology.*

Proposition 2.29 (Proof due to Kedlaya)**.** *Let K be a perfectoid field. If $K^\flat$ is algebraically closed, then K is also.*

Proof. The idea of proof goes as follows.

1. (1) Using completeness, we will get roots by successive approximation.
2. (2) In order to do this approximation, we will find roots in $K^{\flat\circ}$ using the transfer principle of Proposition 2.11
$$K^{\flat\circ}/\varpi^\flat K^{\flat\circ} \cong K^\circ/\varpi K^\circ.$$
Once will get what we want in $K^{\flat\circ}$ will take the untilt of it.

Let $f(T) \in K^\circ[t]$ be a monic polynomial of degree $d \geq 2$. We will construct a sequence $(x_n)_{n\geq 0}$ such that,

1. (1) $|f(x_n)| \leq |\varpi|^n$,
2. (2) and $|x_{n+1} - x_n| \leq |\varpi|^{\frac{d}{n}}$,

which will conclude. To do so, we start very naively. Let $x_0 = 0$. Now we take care of the induction step. Let
$$f(x_n + T) = \sum_{i=0}^{d} b_i T^i.$$
Note that this polynomial is monic with $b_0 = f(x_n)$. If $b_0 = 0$, there is no need to continue. Suppose then that $b_0 \neq 0$. We claim the following.

Claim (1)**.** *We can find $u \in K^\circ$ such that,*

1. *(1)* $\frac{b_i}{b_0} u^i \in K^\circ$.
2. *(2) There is at least $j > 1$ such that* $\frac{b_j}{b_0} u^i \in K^{\circ\times}$.
3. *(3)* $|u| \leq |\varpi|^{\frac{n}{d}}$.

We postpone the proof of this claim to later. Denote then by $g(T)$ any lift of $\sum_{i=0}^{d} \frac{b_i}{b_0} u^i T^i$ *via*
$$K^{\flat\circ}/\varpi^\flat K^{\flat\circ} \cong K^\circ/\varpi K^\circ.$$
We also postpone the proof of the following.

Claim (2)**.** *Let K be a complete and algebraically closed rank 1 valuation field. Let $g \in K^\circ[T]$ be a polynomial of degree at least one such that there exists at least one coefficient (other than the constant coefficient) which is a unit in K°. Then g has a root in K°.*

Note that if the other unit is the leading one, this is immediate because K° is integrally closed.

We continue the proof assuming claim (2). Note that the hypothesis of claim (1) assures that $g(t)$ satifies the hypothesis of claim (2). Let then $y \in K^{\flat\circ}$ with $g(y) = 0$. Now define

$x_{n+1} = x_n + u\sharp(y)$. We have

$$f(x_{n+1}) = f(x_n + u\sharp(y)) = \sum_{i=0}^{d} b_i u^i \sharp(y)^i = b_0 \left(\sum_{i=0}^{d} \frac{b_i}{b_0} u^i \sharp(y)^i \right).$$

Therefore, by construction the expression in parenthesis vanishes modulo ϖ. Then,

(1) $|f(x_{n+1}| \leq |b_0| |\varpi| \leq |\varpi^n| |\varpi| = |\varpi|^{n+1}$,
(2) and $|x_{n+1} - x_n| = |u\sharp(y)| \leq |u| \leq \left|\varpi^{\frac{n}{d}}\right|$.

We now prove both claims.

Proof of claim (1). Note that the value groups of K and $K^\flat$ are isomorphic. As $K^\flat$ is algebraically closed the value group is a $\mathbb{Q}$-vector space. So take an element u such that $|u| = \left|\frac{b_0}{b_j}\right|^{\frac{1}{j}}$ where j is a positive integer such that the following minimum is attained

$$\min\{\left|\frac{b_0}{b_i}\right|^{1/i} \quad b_i \neq 0\}.$$

Note that this minimum is less than $|b_0|^{\frac{1}{d}} \leq |\varpi|^{\frac{n}{d}}$.

Proof of claim (2). First note the following fact : if K is rank 1 valuation field, then any non zero element π of the maximal ideal $K^{\circ\circ}$ is a pseudo-uniformizer. In particular we can take a pseudo-uniformizer π such that g is *monic* with invertible constant coefficient in $R/\pi R$. It follows that,

$$S = R < T > /g(T)$$

is a finite free R-algebra (because it is the case modulo ϖ.) As K is algebraically closed $S_{red}[\frac{1}{\omega}]$ is a product of copies of K. This leads to a map $S \to K$. But as S is finite, it is integrally closed so this factors trough $S \to R$. Now the image of T is a desired root. □

2.3. **Almost toolbox.** This section is dedicated to result of *almost* flavor that are needed to prove almost purity Theorem 2.36. We fix a ϖ-torsion free perfectoid R. We denote by[9]

$$R^{\circ\circ} = \sqrt{\varpi R} = (\varpi^{1/p^\infty}).$$

By Lemma 2.19 we have $(R^{\circ\circ})^2 = R^{\circ\circ}$.

Here are some definitions.

Definition 2.30. Let M be a R-module.

(1) We say that M is *almost zero* if $R^{\circ\circ}M = 0$.
(2) A morphism of R-modules is said to be *almost injective, almost surjective and an almost isomorphism* respectively if the kernel is almost zero, the cokernel is almost zero or both are almost zero.

Remark. Note that a module M is almost zero if and only if for all $n \geq 0$ we have $\varpi^{\frac{1}{p^n}} M = 0$.

Here's the crucial observation.

Lemma 2.31. *The full sub-category of almost zero R-modules is stable by extension and stable by all limits and colmits.*

[9]In the theory of Tate analytic rings, this denotes topologically nilpotent elements, and if we equip R with the ϖ-adic topology, this correspond.

Proof. Only crucial point, for extensions, where $R^{\circ\circ^2} = R^{\circ\circ}$ allows for the argument to go through. □

Note that therefore we can consider the quotient by this sub-category and get again an abelian category.

The most important application of this setup for almost purity is than we can relate almost information on M and almost information on $M/\varpi M$ (see Theorem 2.36). Here is an example of such a phenomenon.

Lemma 2.32. *Let M a ϖ-adically separated R-module. Then M is almost zero if and only if $M/\varpi M$ is almost zero.*

Proof. Suppose that $M/\varpi M$ is almost zero. This means that $R^{\circ\circ}M \subset \varpi M$, and therefore $R^{\circ\circ}M = \varpi M$. Multypliing by $R^{\circ\circ}$ we get $R^{\circ\circ}M = \varpi R^{\circ\circ}M$ (here the crucial property is used). By induction, we conclude

$$R^{\circ\circ}M = \varpi^n R^{\circ\circ}M.$$

□

We now prove some lemmas needed to prove almost purity in the case of fields.

The following definition is not standard but suffices for our needs.

Definition 2.33. Let M be a R-module. Let $\epsilon \in R^{\circ\circ}$. M is *ϵ-almost free of rank d* if there is a map $R^{\oplus d} \to M$ with both kernel and cokernel killed by ϵ.

Here is a again an example of exchange of almost information on R and on $R/\varpi R$.

Lemma 2.34. *Let M be a ϖ-torsion free R-module. Let $n \geq 1$. Then M is $\varpi^{\frac{1}{p^n}}$-almost free of rank d if and only if $M/\varpi M$ is.*

Proof. Note that as $R^{\oplus d}$ is ϖ-torsion free, the map that need to exist in the definition needs to be injective.

First, we suppose that there is a map $R^{\oplus d} \xrightarrow{(m_1,\ldots,m_d)} M$ with cokernel killed by $\varpi^{\frac{1}{p^n}}$. The induced map at ϖ-quotient will also have a cokernel killed by $\varpi^{\frac{1}{p^n}}$. Note that this equivalent to

$$\varpi^{\frac{1}{p^n}} M \subset R(m_1, \ldots, m_d).$$

We show that the kernel of the induced map at ϖ-quotient is killed by $\varpi^{\frac{1}{p^n}}$. Let $(\lambda_i) \in R^{\oplus d}$ be an element of the kernel, so that

$$\sum_{i=0}^{d} \lambda_i m_i = \varpi m'$$

Multiply this by $\varpi^{\frac{1}{p^n}}$.

$$\sum_{i=0}^{d} \varpi^{\frac{1}{p^n}} \lambda_i m_i = \varpi \varpi^{\frac{1}{p^n}} m'$$

Therefore let $(\mu_i) \in R^{\oplus d}$ such that

$$\varpi^{\frac{1}{p^n}} m' = \sum_i \mu_i m_i.$$

Therefore,

$$\sum_i (\varpi^{\frac{1}{p^n}} \lambda_i - \varpi \mu_i) m_i = 0.$$

Using injectivity, we see that $\varpi^{\frac{1}{p^n}}(\lambda_i) = \varpi(\mu_i)$ which concludes.

For the other direction, suppose that there is a map

$$(R/\varpi R)^{\oplus d} \xrightarrow{(\overline{m_1}, \ldots, \overline{m_d})} M/\varpi M,$$

with kernel and cokernel both killed by $\varpi^{\frac{1}{p^n}}$. Take any lift of the map

$$R^{\oplus d} \xrightarrow{(m_1, \ldots, m_d)} M.$$

As

$$R(m_1 \ldots, m_d) \subset \varpi^{\frac{1}{p^n}} M + \varpi M \subset \varpi^{\frac{1}{p^n}} M,$$

the corkenel of this map is killed by $\varpi^{\frac{1}{p^n}}$. We now show that this map is injective to conclude the proof. In what follows we denote by $\pi = \varpi^{\frac{1}{p^n}}$. Let (λ_i) be in the kernel. By hypothesis, there exists (μ_i) such that $\pi(\lambda_i) = \pi^{p^n}(\mu_i)$. So that $(\lambda_i) \in \pi^{p^n - 1} R^{\oplus d}$. As M is ϖ torsion-free, it also follows that (μ_i) is in the kernel. Therefore, by induction for every $k \geq 1$,

$$(\lambda_i) \in \pi^{kp^n - k} R^{\oplus d}.$$

As $n \geq 1$, this concludes. □

The following lemma is key to the proof of almost purity.

Lemma 2.35. *Let K be a perfectoid field of characteristic p and L be a finite field extension. Let $n \geq 0$. Then the $\mathcal{O}_K$-module $\mathcal{O}_L$ is $\varpi^{\frac{1}{p^n}}$-almost free of rank $[L : K]$.*

Proof. Denote by d the rank of the extension. Note that L is a separable extension because K is perfect. Therefore the trace pairing is non-degenerate. In particular for $K \to L$, let $e = \sum_{i=0}^d e_i \otimes e_i^* \in L \otimes_K L$ be the canonical element corresponding to the trace. Here is the crucial calculation. Let N be enough such that $\varpi^N e_i, \varpi^N e_i^* \in \mathcal{O}_L$. As $\mathcal{O}_K$ is perfect, it follows that $\varpi^{N/p^n} \in \mathcal{O}_K$. Therefore for every $b \in \mathcal{O}_L$ we get that,

$$\varpi^{2N/p^n} b = \sum_{i=0}^d \varpi^{N/p^n} e_i \operatorname{Tr}(\varpi^{N/p^n} e_i^* b).$$

It now follows that the map,

$$\mathcal{O}_K^{\oplus d} \xrightarrow{(\varpi^{N/p^n} e_i)} \mathcal{O}_L$$

is injective of cokernel killed by $\varpi^{\frac{2N}{p^n}}$. □

2.4. **Almost purity for fields.** We prove in this section almost purity in the case of perfectoid fields.

Theorem 2.36 (Almost purity – point case.)**.** *Let K be a perfectoid field. First of all if L is a finite extension over K, then it is a perfectoid field. Then, the tilt functor is a degree preserving equivalence of caetgories,*

$$\{\textit{Finite extensions of } K\} \to \{\textit{Finite extensions of } K^\flat\}.$$

Therefore, fixing an algebraic closure on both sides (so a colimit of all finite extensions) will canonically produce one on the other. Therefore up to fixing an algebraic closure on one side,

$$\mathrm{Gal}(\overline{K}, K) \cong \mathrm{Gal}(\overline{K^\flat}, K^\flat).$$

Proof. Note that in characteristic p seeing that if L is a finite extension of $K^\flat$ then L is perfectoid is easy. By elementary theory of valuations, there exist a unique valuation on L that extends the one on K and for which L is complete (and necessarily discretely valued as K is).

Also, we already know by Proposition 2.27 that we have an equivalence of categories

$$\{\text{Perfectoid fields over } K\} \to \{\text{Perfectoid fields over } K^\flat\},$$

but we know nothing about how it preseves the finiteness.

Claim (1)**.** *The untilt functor restricts and factors to*

$$\{\textit{Finite perfectoid fields over } K^\flat\} \to \{\textit{Finite fields over } K\}$$

Proof of claim (1). Let L be a finite extension of $K^\flat$, say of degree d.

- By lemma 2.35, $\mathcal{O}_L$ is almost finite free of degree d over $\mathcal{O}_{K^\flat}$.
- By lemma 2.34, $\mathcal{O}_L/\varpi^\flat\mathcal{O}_L$ is almost finite free of degree d over $\mathcal{O}_{K^\flat}/\varpi^\flat\mathcal{O}_{K^\flat}$.
- By the $\sharp$ isomorphism of Proposition 2.11, $\mathcal{O}_{L^\sharp}/\varpi\mathcal{O}_{L^\sharp}$ is almost finite free of degree d over $\mathcal{O}_K/\varpi\mathcal{O}_K$.
- By lemma 2.34, $\mathcal{O}_{L^\sharp}$ is almost finite free of degree d over $\mathcal{O}_K$.
- So it follows that $L^\sharp$ is a finite K module by inverting ϖ.

We are left to show that the following composition

$$\{\text{Finite fields over } K^\flat\} \xrightarrow{\sharp} \{\text{Finite fields over } K \text{ which are perfectoid}\} \subseteq \{\text{Finite fields over } K\},$$

is essentially surjective. To this end, let E be the completion of the algebraic closure of $K^\flat$. Note that this is algebraically closed by Krasner's lemma. Therefore by proposition 2.29 $N = E^\sharp$ is also algebraically closed. Let $M = \bigcup L^\sharp \subset N$ for L finite over $K^\flat$. Note that M is dense in N because on the level of ring of integers the inclusion is an isomorphism modulo ϖ. This can be seen commuting comilits and using the untilt isomorphism of Proposition 2.11. By Krassner's lemma, M is algebraically closed.

Now if E is a finite extension of K, as $K \subset M$ is an algebraically closed extension, we have $E \subset M$, so $E \subset L^\sharp$ for a finite Galois extension L of K. As the $\sharp$ functor is fully-faithful and the untilt is degree preserving $L^\sharp$ is also Galois. So as the untilt is fully faithufl, by cardinality on isomorphism classes the following composition is essentially surjective

$$\begin{aligned}\{\text{Sub-extensions of } L \text{ over } K^\flat\} \xrightarrow{\sharp} \{&\text{Sub-extensions of } L^\sharp \text{ over } K \text{ which are perfectoid}\}\\ &\subseteq \{\text{Sub-extensions of } L^\sharp \text{ over } K\},\end{aligned}$$

and this concludes the proof. □

We mention the following general case.

Theorem 2.37 (Almost purity – general case)**.** *Let (A, A^+) be a perfectoid Huber pair. First of all if (B, B^+) is finite étale over (A, A^+), then it is a perfectoid Huber pair. Then, we have the following equivalences of categories*

$$\begin{tikzcd}
\{\textit{Finite étale } (A, A^+)\textit{-algebras}\} \arrow[r, "\flat"] & \{\textit{Finite étale } (A^\flat, A^{+\flat})\textit{-algebras}\} \\
\{\textit{Almost finite étale } A^+\textit{-algebras}\} \arrow[u, "{[\frac{1}{\varpi}]}"] \arrow[d, "\bmod \varpi"'] & \{\textit{Almost finite étale } A^+\textit{-algebras}\} \arrow[u, "{[\frac{1}{\varpi^\flat}]}"'] \arrow[d, "\bmod \varpi^\flat"] \\
\{\textit{Almost finite étale } A^+/\varpi A^+\textit{-algebras}\} \arrow[r, "\sim"] & \{\textit{Almost finite étale } A^{+\flat}/\varpi A^+/\varpi^\flat A^{+\flat}\textit{-algebras}\}
\end{tikzcd}$$

Let X be a perfectoid space. Then $\flat : X \to X^\flat$ induces an equivalence of étale sites

$$X_{et} \cong X^\flat_{et}.$$

Appendix A. Miscellany on completeness

A.1. **Classical.** For the remaining of this section R is a ring and I is a *finitely generated module.*

We essentially re-arrange and rephrase lemmas from [Sta24, Section 00M9] and [GR18, 8.2 and 8.3].

Definition A.1 (I-adic topology, completeness)**.** Let M be a R-module. The I-adic topology on M is the topology turning M into a topological abelian group with subgroups $(I^n M)_{n\in N}$ being open neighbourhoods of zero. Let $(x_n), (y_n) \in M^{\mathbb{N}}$ be sequences.

(1) We say that a sequence (x_n) *converges to zero* if for every $N \in \mathbb{N}$ there is n_0 such that for every $n \geq n_0$ we have $x_n \in I^N M$.
(2) We say that (x_n) is *Cauchy* if for every $N \in \mathbb{N}$ there is some n_0 such that for every $n, m \geq n_0$ such that $x_n - x_m \in I^N M$.
(3) We say that two Cauchy sequences (x_n) and (y_n) are *equivalent* if $(x_n - y_n)$ converges to zero.
(4) We say that (x_n) converges to x if (x_n) is equivalent to the constant sequence with value x.
(5) We say that M is *I-adically complete* if every Cauchy sequence has a unique limit in M.

Remark. Equipping R with the I-adic topology turns M with the I-adic topology into a topological R-module.

Remark. Topological abelian groups always have a *completion* $M \to M^\wedge$ realized has the quotient of the abelian group of Cauchy sequences by the subgroup of Cauchy sequences equivalent to zero (see [GR18, Theorem 8.2.8]). When the topology is I-adic, the completion can be realized as an inverse limit as shows the next lemma.

Lemma A.2 (I-adic completion)**.** *Let M be a R-module. Equip it with the I-adic topology. Then there is an homeomorphism of topological R-modules*

$$M^\wedge \cong \varprojlim_n M/I^n M$$

where the topology on the right is given by the inverse limit topology which each component being equipped with the discrete topology.

Proof. We first define a compatible collection of continuous maps $p_n \colon M^\wedge \to M/I^n M$. Given a Cauchy sequence (x_k), note that $x_k \mod I^n$ does not depend on k for large enough k, so we can define this to be p_n. Also, it is clear that two equivalent sequences give the same value on p_n. This is another way of saying that we extend by the completeness property the continuous map $M \to M/I^n M$.

We construct an inverse map. Given an element $(\overline{m_n}) \in \varprojlim_n M/I^n M$, any lift $(m_n) \in M^{\mathbb{N}}$ is a Cauchy sequence. Also, we see that two different lifts give equivalent Cauchy sequences. Convergent sequences in $\varprojlim_n M/I^n M$ are eventually constant sequences, and the continuity follows from checking that convergent sequences are sent to convergent sequences. $\square$

Remark. We can therefore treat I-adic completeness for a module M as being the property that the natural map $M \to \varprojlim_n M/I^n M$ is an isomorphism.

Definition A.3 (I-adic completion)**.** Let M be a R-module. The I-adic completion is defined to be

$$M^{\wedge,I} = \varprojlim_n M/I^n M.$$

The natural topology to consider is the topology of the inverse limit. By Lemma A.2 the I-adic completion is the completion of M if we equip M with the I-adic topology.

Lemma A.4 (Preserving surjections)**.** *Let $\varphi\colon M \to N$ be a map of R-modules such that $M/IM \to N/IN$ is surjective, then $M^{\wedge,I} \to N^{\wedge,I}$ is surjective. In particular this holds if $M \to N$ is surjective.*

Proof. Note first that $M/I^n M \to N/I^n N$ is surjective using Nakayama with the nilpotent ideal I in R/I^n. Say

$$K_n = \ker(M/I^n M \to N/I^n N).$$

It suffices to show that the natural map $K_{n+1} \to K_n$ is surjective implying that $R^1 \lim K_n = 0$ and therefore the claimed surjectivity. Let $x \in M$ be a lift of an element in K_n. Therefore $\varphi(x) \in I^n N$. Note that $N = \varphi(M) + IN$ by assumption. So $\varphi(x) \in \varphi(I^n M) + I^{n+1}N$. But then there is $y \in I^n M$ such that $\varphi(x - y) \in I^{n+1}N$, which concludes. □

The following holds when I is finitely generated.

Lemma A.5 (I-adic completion is I-adically complete.)**.** *Let M be a module. Then the topology on $M^{\wedge,I}$ is the I-adic topology. In particular $M^{\wedge,I}$ is I-adically complete.*

Proof. Using Lemma A.2, it suffices to prove that $I^n(M^{\wedge,I}) = \ker(M^{\wedge,I} \to M/I^n M)$. Because we suppose that I is finitely generated, we can construct a surjection $M^{\oplus k} \to I^n M$ that leads to a surjection

$$(M^{\wedge,I})^{\oplus r} \to (I^n M)^{\wedge,I} = \varprojlim_m I^n M/I^{n+m}M = \ker(M^{\wedge,I} \to M/I^n M).$$

But the image of this surjection is exactly $I^n(M^{\wedge,I})$. □

Lemma A.6 (Units and completion)**.** *Let R be I-adically complete. Then r is a unit if and only if it is a unit modulo I.*

Proof. For the non-trivial direction, r will be a unit in R/I^n for all n if it is a unit in R/I. But then as $R = \varprojlim_n R/I^n$ we see that r is component-wise invertible in $\prod_n R/I^n$ which concludes. □

Lemma A.7 (Completeness is preserved under finer adic topologies)**.** *Let $I \subset J$ be ideals of R. Let M be an R-module. If M is J-adically complete, then M is also I-adically complete.*

Proof. Note that has M is J-adically separated, M is also I-adically separated. Let then (x_n) be a Cauchy sequence for the I-adic topology. Let x be the limit in the J-adic topology.

Say $I = (f)$. Up to taking a subsequence we can suppose that $x_n - x_{n+1} \in I^n$. Therefore we can write

$$x_n - x_{n+1} = f^n z_n.$$

But then for any $m \geq n$ we have

$$x_n - x_m = \sum_{k=n}^{m-1}(x_k - x_{k+1}) = \sum_{k=n}^{m-1} f^n z_n = f^n(\sum_{k=n}^{m-1} z_n f^{k-n}).$$

We can take the limit in m for the J-adic topology of the above to get

$$x_n - x = f^n y_n$$

for some y_n. It shows the claim when I is generated by one element.

Now, note that if $I = (f_1, \dots, f_r)$ then a basis of neighborhoods of zero is given by ideals $(f_1^n, \dots, f_r^n)$ when n varies. So in the above, up to taking again a subsequence, we can suppose that

$$x_n - x_{n+1} = \sum_{i=1}^{r} f_i^n z_{i,n}.$$

So one can redo the same proof, up to writing an additional sum. □

Definition A.8. Let M be a R-module. We equip M with the I-adic topology. We say that a submodule $N \subset M$ is *Artin-Ress* if the a priori finer I-adic topology on N agrees with the subspace topology. This is equivalent to the following. For every n there is some $N \geq n$ with the property that

$$(I^N M) \cap N \subset I^n M.$$

See [Sta24, Lemma 00IN] or [GR18, Theorem 11.4.46] for conditions where this holds.

Lemma A.9 (Exactness of I-adic completion)**.** *Let M be an R-module and $N \subset M$ an Artin-Rees submodule. Then*

$$0 \to N^{\wedge,I} \to M^{\wedge,I} \to (M/N)^{\wedge,I} \to 0$$

is exact.

Proof. Because we suppose that the I-adic topology on N is the subspace topology, the I-adic completion correspond to the completion with respect to the subspace topology on N. Note that the quotient topology on M/N is the I-adic topology. Therefore, we are asked to show that the completion functor if we equip N and M/N with their natural topologies is exact. Note that by Lemma A.4 the right arrow is surjective.

We use to this effect the description of the completion as the quotient of Cauchy sequences. That the first map is injective is easily seen with this perspective, using again that we equip N with the subspace topology. We are left to show that the map is exact in the middle. Take a Cauchy sequence $(\pi(m_n)) \in (M/N)^{\mathbb{N}}$ if $\pi\colon M \to N$ denotes the quotient map. Then this means that for any k there is some $n(k)$ such that we have $m_{n(k)} \in N + I^k M$. So take $h_{n(k)} \in N$ such that $m_{n(k)} - h_{n(k)} \in I^k M$. It follows that we have Cauchy sequences $(h_{n(k)})$ and $(m_{n(k)})$ in k which are equivalent by construction and with $(h_{n(k)})$ being Cauchy in N. □

Lemma A.10 (Completeness and Witt vectors)**.** *Let R be a perfect algebra which is ϖ-complete for some $\varpi \in R$.*

(1) Rings $W_n(R)$ are $(p, [\varpi])$-complete.
(2) The ring $W(R)$ is $(p, [\varpi])$-complete.

Proof. For item (1), as p is nilpotent, we actually need to show that $W_n(R)$ is $[\varpi]$-complete. We equip $W_n(R)$ with the $[\varpi]$-topology and proceed by induction with the case $n = 1$ being an hypothesis.

Note that $R \cong V_n(R)/V_{n+1}(R)$ is an Artin-Rees submodule. Indeed being in $V_n(R)/V_{n+1}(R)$ means that the first component n-th components are zero. But if the first n-th components of $[\varpi]^k(x_i)$ are zero then $[\varpi]^k(x_i) = [\varpi]^k(0, 0, \ldots, x_n)$. Also, this sub-module is isomorphic as a topological R-module to R. But now the induction step goes using a five lemma argument involving the exact sequences of Lemma 1.14 and Lemma A.9.

For item (2), using item (1), we see that $W(R) = \varprojlim_n W_n(R)$, with the limit topology with the topology on each factor being the $(p, [\varpi])$-topology, is complete. Let $\pi_n \colon W(R) \to W_n(R)$ the projection. The aforementionened limit topology is given by the system of neighborhoods of zero (indexed by n and k) $\pi_n^{-1}([\varpi]^k) = (p^n, [\varpi]^k)$, which concludes. □

A.2. **Derived.** Derived complete modules have better homological properties (Lemma A.21) and fit into the local duality philosophy (Theorem A.14). There is also a tight connection between the classical and the derived notions which we highlight (Lemmas A.17 and A.19).

A.2.1. *Local duality yoga and derived completions.* References for this section are [BHV18, Section 2 and 3], [HPS97], [Sta24, Tag 091N], [Sta24, Tag 0A6V], [DG00] and [BS14, Section 3.4].

Let R be any ring. We consider $\mathcal{D}(R)$ the stable ∞-category of R-modules. We also fix a finitely generated ideal together with generators $I = (f_1, \ldots, f_n)$.

Definition A.11 (Koszul complexes, [Sta24, Section 0621])**.** Let $f \in R$. We define $\mathrm{Kos}^1(f) = \mathrm{fib}(R \xrightarrow{r} R)$. Therefore, seen as a complex

$$\mathrm{Kos}^1(f) = (R \xrightarrow{f} R)$$

with the source in degree zero. Let $f_1, \ldots, f_r \in R$. We define

$$\mathrm{Kos}^1(f_1, \ldots, f_r) = \bigotimes_{i=1}^{r} \mathrm{Kos}(f_i).$$

As a complex, this can be seen as the exterior algebra $\bigwedge R^n$ with differentials given by multiplication by the f_i along the simplicial identities. This can be seen as coming from the tensor product in the category of chain complexes. We define

$$\mathrm{Kos}^n(f_1, \ldots, f_r) = \mathrm{Kos}^1(f_1^n, \ldots, f_r^n).$$

We define $\mathrm{Kos}^\infty(f_1, \ldots, f_r) = \varinjlim_n \mathrm{Kos}^n(f_1, \ldots, f_r)$.

Remark. Note that $\mathrm{Kos}^\infty(f) = \mathrm{fib}(R \to R[f^{-1}])$, because $R[f^{-1}] = \varinjlim(R \xrightarrow{f} R \xrightarrow{f} \cdots)$. Extending on this argument one can see ([Sta24, Tag 0913]) that

$$\mathrm{Kos}^\infty(f_1, \ldots, f_r) = \mathrm{fib}(R \to \check{C}(f_1, \ldots, f_r))$$

where $\check{C}(f_1, \ldots, f_r))$ is

$$\prod_i R_{f_i} \to \prod_{i<j} R_{f_i f_j} \to \cdots \to R_{f_1 \cdots f_r}$$

the alternating Čech complex of R for the cover of $U = \mathrm{Spec}(R) \setminus V(f_1, \dots, f_r) = \bigcup_{i=1}^{n} D(f_i)$. But this last complex is $R\Gamma(U, \mathcal{O}_U)$, because it is quasi-isomorphic to the usual Čech complex, see [Sta24, Tag 01FM]. Therefore we deduce a fiber sequence

$$\mathrm{Kos}^{\infty}(f_1, \dots, f_r) \to R \to R\Gamma(U, \mathcal{O}_U).$$

In consequence, we see that $\mathrm{Kos}^{\infty}(f_1, \dots, f_r)$ only depends on the *topological closed subset* $V(f_1, \dots, f_r)$. See next definition.

Definition A.12 (Local cohomology)**.** Let R be a ring and I be an ideal and denote by $U = \mathrm{Spec}(R) \setminus V(I)$. We define

$$R\Gamma_I(R) = \mathrm{fib}(R \to R\Gamma(U, \mathcal{O}))$$

and call it the I-local cohomology of R. In the above setup, if $I = (f_1, \dots, f_r)$, we see that

$$R\Gamma_I(R) = \mathrm{Kos}^{\infty}(f_1, \dots, f_r).$$

As a complex, this can be therefore seen as the *augmented Čech complex*

$$R \to \prod_i R_{f_i} \to \prod_{ij} R_{f_i f_j} \to \cdots \to R_{f_1 \cdots f_r}$$

with R in degree zero.

Remark. Note that localizing at every $f \in I$ yields an isomorphism $R_f \to R\Gamma(U, \mathcal{O})_f = R\Gamma(D(f), \mathcal{O})$ implying that $R\Gamma_I(R)$ is I-torsion. In fact, this is the universal torsion object mapping to R, as we will soon explain.

Definition A.13 (Left orthogonals)**.** Let $\mathcal{C}$ be a stable ∞-category. Let $\mathcal{D} \subseteq \mathcal{C}$ be a full-subcategory of $\mathcal{C}$. The *left orthogonal* $\mathcal{D}^{\perp}$ is defined as the full subcategory whose objects $Y \in \mathcal{C}$ are those satisfying $\mathrm{Hom}(X, Y) = 0$ for any $X \in \mathcal{D}$. Here Hom denotes the spectrally enriched mapping space. This is the object underlying $\mathrm{R}\,\mathrm{Hom}(X, Y)$ in the homotopy category.

We define now various sub-stable ∞-categories of $\mathcal{D}(R)$.

(1) The full subcategory $\mathcal{D}(R)_{I-tors}$ such that homotopy groups are I^n-torsion for some n. This is seen to be the smallest localizing sub-category containing the perfect complex $\mathrm{Kos}^1(I) = \mathrm{Kos}^1(f_1, \dots, f_n)$. It is therefore presentable and compactly generated by one object.
(2) The category $\mathcal{D}(R)_{U-loc}$ of $U = \mathrm{Spec}(R) \setminus V(I)$-local objects is defined to be the *left orthogonal of* $\mathcal{D}(R)_{I-tors}$.
(3) The category $\mathcal{D}(R)_{I-comp}$ of I-complete objects is defined to be the *left orthogonal of* $\mathcal{D}(R)_{U-loc}$.

It follows formally that these categories are well behaved, first of all stable, but also inclusion into $\mathcal{D}(R)$ will be adjunctions with good properties by the adjoint functor theorem [HTT, Cor. 5.5.2.9] and the structure of left orthogonal sub-categories [HPS97, Theorem 3.3.5]. We resume all these properties in the following Theorem.

Theorem A.14 (Local duality)**.** *Let R be a ring, and I a finitely generated ideal.*

(1) The three sub-stable ∞-categories $\mathcal{D}(R)_{I-tors}$, $\mathcal{D}(R)_{U-loc}$, $\mathcal{D}(R)_{I-comp}$ arrange themselves in the following diagram of adjunctions

$$\begin{tikzcd}
 & \mathcal{D}(R)_{U-loc} \arrow[d, shift left] & \\
 & \mathcal{D}(R) \arrow[u, shift left, "R\Gamma_U"] \arrow[dl, shift left, "R\Gamma_I"] \arrow[dr, shift left, "\Lambda_I"] & \\
\mathcal{D}(R)_{I-tors} \arrow[ur, shift left] & & \mathcal{D}(R)_{I-comp} \arrow[ul, shift left]
\end{tikzcd}$$

*where $R\Gamma_I$ the I-*local cohomology *is a right adjoint, and $R\Gamma_U$ and Λ_I the* cohomology in U *and the* derived completion *are left adjoints.*

(2) We have

$$\ker(R\Gamma_U) = \mathcal{D}(R)_{I-tors} \quad \ker(\Lambda_I) = \ker(R\Gamma_I) = \mathcal{D}(R)_{U-loc}$$

(3) Functors $R\Gamma_I$ and $R\Gamma_U$ are smashing*, meaning that for $M \in \mathcal{D}(R)$*

$$R\Gamma_I(M) = M \otimes R\Gamma_I(R) \quad R\Gamma_U(M) = M \otimes R\Gamma_U(R),$$

and $R\Gamma_I(R)$ and $R\Gamma_U(R)$ are described by the complexes defined above, so the augmented Čech complex *and the* Čech complex *respectively. In consequence we have the fiber sequence*

$$R\Gamma_I(M) \to M \to R\Gamma_U(M).$$

(4) We have natural equivalences $R\Gamma_I\Lambda_I = R\Gamma_I$ and $\Lambda_I R\Gamma_I = \Lambda_I$. This means that local cohomology does not see completion and that completion does not see local cohomology.

(5) Functors

$$R\Gamma_I \colon \mathcal{D}(R)_{I-comp} \to \mathcal{D}(R)_{I-tors} \quad \Lambda_I \colon \mathcal{D}(R)_{I-tors} \to \mathcal{D}(R)_{I-comp}$$

are mutually inverse equivalences of categories.

(6) If $\underline{\mathrm{Hom}}$ denotes the enriched internal Hom*, then for any $M, N \in \mathcal{D}(R)$ we have*

$$\underline{\mathrm{Hom}}(R\Gamma_I M, N) = \underline{\mathrm{Hom}}(M, \Lambda_I N)$$

in particular $\Lambda_I N = \underline{\mathrm{Hom}}(R\Gamma_I(R), N)$. Therefore one can use the augmented Čech complex *to compute the derived completion.*

(7) If $I = (f_1, \ldots, f_r)$ and $R/^L(f_1^n, \ldots f_r^n)$ denotes the derived quotient ring*, then*

$$\Lambda_I N = \varprojlim_n N \otimes R/^L(f_1^n, \ldots f_r^n).$$

Proof. Items (1)-(2)-(3) are formal consequences of the adjoint functor theorem [HTT, Cor. 5.5.2.9] and the structure of left orthogonal sub-categories [HPS97, Theorem 3.3.5]. Let us explain how we get the consequences. To get (4), apply Λ_I to

$$R\Gamma_I(M) \to M \to R\Gamma_U(M)$$

and (2) to get $R\Gamma_I\Lambda_I = R\Gamma_I$. Note also that the left term in

$$\mathrm{fib}(M \to \Lambda_I M) \to M \to \Lambda_I M$$

is U-local, so applying $R\Gamma_I$ gets the second claim. Item (5) follows from (4) using that $R\Gamma_I$ and Λ_I are naturally isomorphic to identities functors on $\mathcal{D}(R)_{I-tors}$ and $\mathcal{D}(R)_{I-comp}$ respectively.

Item (6) is an adjunction play-around together with (5), as shown below.

$$\begin{aligned}\underline{\mathrm{Hom}}(R\Gamma_I M, N) &\cong \underline{\mathrm{Hom}}(R\Gamma_I M, R\Gamma_I N)\\ &\cong \underline{\mathrm{Hom}}(\Lambda_I R\Gamma_I M, \Lambda_I R\Gamma_I N)\\ &\cong \underline{\mathrm{Hom}}(\Lambda_I M, \Lambda_I N)\\ &\cong \underline{\mathrm{Hom}}(M, \Lambda_I N).\end{aligned}$$

For item (7), we clarify what we mean by *derived quotient ring*. Namely we look at R as a $\mathbb{Z}[x_1, \dots, x_r]$ algebra sending $x_i \mapsto f_i$ and we consider the animated ring

$$R \otimes^L_{\mathbb{Z}[x_1,\dots,x_r]} \mathbb{Z}[x_1, \dots, x_r]/(x_1^n, \dots, x_r^n).$$

Note that has a complex, we can compute this tensor product using a projective resolution of $\mathbb{Z}[x_1, \dots, x_n]/(x_1^n, \dots, x_r^n)$. Because $(x_1^n, \dots, x_r^n)$ is regular, we can take as a resolution

$$\mathrm{Kos}_n(x_1, \dots, x_r) = \Sigma^r \mathrm{Kos}^n(x_1, \dots, x_r) = \underline{\mathrm{Hom}}(\mathrm{Kos}^n(f_1, \dots, f_r), R).$$

The shift is here so that the last non-zero term is in degree zero, so that is it is indeed the desired projective resolution. Therefore, as a complex, the considered animated ring is

$$\mathrm{Kos}_n(f_1, \dots, f_r) = R/^L(f_1^n, \dots f_r^n).$$

We now proceed to the proof of the statement. We have

$$\begin{aligned}\Lambda_I N &\cong \underline{\mathrm{Hom}}(R\Gamma_I(R), N)\\ &\cong \underline{\mathrm{Hom}}(R\Gamma_I(R), N)\\ &\cong \underline{\mathrm{Hom}}(\varinjlim_n \mathrm{Kos}^n(f_1, \dots, f_r), N)\\ &\cong \varprojlim_n \underline{\mathrm{Hom}}(\mathrm{Kos}^n(f_1, \dots, f_r), N)\\ &\cong \varprojlim_n \underline{\mathrm{Hom}}(\mathrm{Kos}^n(f_1, \dots, f_r), R) \otimes N\\ &\cong \varprojlim_n \mathrm{Kos}_n(f_1, \dots, f_r) \otimes N.\end{aligned}$$

□

Example A.15. Let A be a Noetherian local ring, let K denotes it's fraction field. Then as $K/A = R\Gamma_{\mathfrak{m}}(A)[1]$ we have

$$\Lambda_{\mathfrak{m}} K/A = (\Lambda_{\mathfrak{m}} R\Gamma_{\mathfrak{m}}(A))[1] = \Lambda_{\mathfrak{m}} A[1] = A^{\mathfrak{m},\wedge}[1]$$

where the equality with the classical completion follows from Lemma A.17.

A.2.2. *Properties of derived completion.* We now inspect how derived completion behaves when we start from ordinary modules.

Definition A.16 (Derived complete module)**.** We say that an R-module M is *derived I-complete* if $M \in \mathcal{D}(R)_{I-comp}$, meaning that it is derived complete when seen as an object of the derived category.

Remark. A module M is I-derived complete if and only if $\mathrm{Hom}_R(R_f, M) = \mathrm{Ext}^1_R(R_f, M) = 0$ for every $f \in I$ because R_f has a projective resolution of length 2 writing the direct colimit as a coequalizer, and that R_f for $f \in I$ are generators for U-local objects.

Remark. Let us expand on Theorem A.14 to understand more how to compute derived completion. Say N is a discrete module. We have seen in the proof that

$$\Lambda_I N = \varprojlim_n \underline{\mathrm{Hom}}(\mathrm{Kos}_n(f_1, \ldots, f_r), N).$$

Because $\mathrm{Kos}_n(f_1, \ldots, f_r)$ can be seen as a complex of projective modules, then cohomology of $\underline{\mathrm{Hom}}(\mathrm{Kos}_n(f_1, \ldots, f_r), N)$ can be computed has the homology of $\mathrm{Kos}_n(N; f_1, \ldots, f_r) = N \otimes^L R/^L(f_1^n, \ldots, f_r^n)$ which is a complex concentrated in homological degree $[0, r]$. Write N_n for this complex. Then, from the usual sequences for derived inverse limits ([Sta24, Tag 0CQE]) we have exact sequences for every $k \in \mathbb{Z}$,

$$0 \to R^1 \varprojlim_n H^{k-1}(N_n) \to H^k(\Lambda_I N) \to \varprojlim_n H^k(N_n) \to 0.$$

This is a tool to compute homotopy groups of the derived completion.

In particular for $k = 0$ we get

$$0 \to R^1 \varprojlim_n H^{-1}(N_n) \to H^0(\Lambda_I N) \to \varprojlim_n M/(f_1^n, \ldots, f_r^n) \to 0,$$

seeing that the $\pi_0(\Lambda_I M)$ always surject to the classical completion.

Note that for $k > 0$ we have $H^k(N_n) = 0$ because N_n is connective. Also for $k > 0$ the system $(H^{k-1}(N_n))_n$ is Mittag-Leffler because it is either identically zero or the system $(M/(f_1^n, \ldots, f_r^n))_n$ when $k = 1$. Therefore we deduce that $\Lambda_I N$ is connective by vanishing of $R^1 \varprojlim_n$ in this range. Note also that for $k < -r$ both extremities of the exact sequence are also zero because N_n is concentrated in homological degrees $[0, r]$, concluding therefore that $\Lambda_I N$ is concentrated in homological degrees $[0, r]$.

Remark. We expand on the last remark. Namely we treat the case where $I = (f)$ is principal. In this case

$$\mathrm{Kos}_n(M; f) = (M \xrightarrow{f^n} M)$$

with the target in degree zero. The system where we want to take the derived inverse limit on is

$$\begin{tikzcd} M \arrow[r, "f^{n+1}"] \arrow[d, "f"] & M \arrow[d, "="] \\ M \arrow[r, "f^n"] & M \end{tikzcd}$$

In particular we see that $\pi_1(\mathrm{Kos}_n(M; f)) = M[f^n]$ the f^n-torsion of M. The inverse system on π_1 is given by the maps $(\cdots M[f^{n+1}] \xrightarrow{f} M[f^n] \to \cdots)$. We have two pertinent exact sequences in this case; one gives

$$0 \to R^1 \varprojlim_n M[f^n] \to \pi_0(\Lambda_I N) \to \varprojlim_n M/f^n \to 0,$$

while the other yields $\pi_1(\Lambda_I N) = \varprojlim_n M[f^n]$. This description is enough to get our first important result.

Lemma A.17 (Classical vs. derived (1))**.** *Let N be a discrete module with bounded f^∞-torsion, meaning that the sequence of submodules*

$$M[f] \subset M[f^2] \subset \cdots M[f^n] \subset \cdots$$

stabilizes. Then the derived f-completion of M is discrete and coincides with the classical f-adic completion. In particular, this applies if the module is f-torsion free or if M is Noetherian.

Proof. Up to replacing f with a power of f we can suppose that $M[f] = M[f^2]$. Therefore the system $\cdots \to M[f] \xrightarrow{f} M[f])$ has all arrows being zero, implying that $0 = \varprojlim_n M[f^n] = R^1 \varprojlim_n M[f^n]$, giving the claim following the above discussion. □

We also note the following case where the torsion is really tame.

Lemma A.18 (Bounded torsion in perfect algebras)**.** *Let B be perfect in characteristic p and $b \in B$. Then for any $m, n \in \mathbb{N}$ all the inclusions*

$$B[b^{1/p^m}] \subset B[b] \subset B[b^{p^n}]$$

are equalities. In particular B has bounded b-torsion, and the derived b-completion coincides with the classical completion.

Proof. Reversing the first inclusion is sufficient. Say $bx = 0$. Then $b^{1/p^m} x^{1/p^m} = 0$. But then $b^{1/p^m} x = b^{1/p^m} x^{1/p^m} x^{1-1/p^m} = 0$. This concludes. □

Lemma A.19 (Classical vs. derived (2))**.** *An R-module M is classically complete if and only if it is I-adically separated and derived complete.*

More precisely, this is deduced by proving that every Cauchy sequence has a limit in a derived complete module.

Proof. First, note that M/I^n is derived I-complete. Indeed we only need to check that $R\varprojlim_n(\cdots \xrightarrow{f} M/I^n \xrightarrow{f} M/I^n) = \underline{\mathrm{Hom}}(R_f, M/I^n) = 0$ for any $f \in I$. But as f acts nilpotently on M/I^n, this concludes.

Now, if M is classically I-complete, then as $M = \varprojlim_n M/I^nM = R\varprojlim_n M/I^nM$ and that derived complete modules are stable under homotopy limits, we deduce that M is derived I-complete.

Suppose now that M is derived I-complete. But then $\pi_0(\Lambda_I M) = M \to \varprojlim_n M/I^n$ is surjective, as explained above. □

Lemma A.20. *An object $M \in \mathcal{D}(R)$ is I-derived complete if and only if $\pi_i(M)$ is a derived I-complete module for every $i \in \mathbb{Z}$.*

Proof. Let $f \in I$. Note that A_f has a projective resolution of length 2, writing the colimit as a coequalizer. Therefore we have a spectral sequence degenerating at the E_2 page with only two columns

$$\mathrm{Ext}^p_R(A_f, H^q(M)) \implies \mathrm{Ext}^{p+q}(A_f, M).$$

Therefore we have exact sequences that gives us the conclusion

$$0 \to \mathrm{Ext}^1_R(R_f, H^{p-1}(M)) \to \mathrm{Ext}^p_R(R_f, M) \to \mathrm{Hom}_R(R_f, H^p(M)).$$

□

Lemma A.21 (Derived complete modules are abelian)**.** *The full subcategory of* Mod_R *consisting of derived* I*-complete modules is an abelian subcategory closed under products, kernels, cokernels and extensions. Moreover* $M \to \pi_0(\Lambda_I M)$ *is left adjoint to the inclusion of derived complete modules into* R*-modules.*

Proof. First, note that products are homotopy products so this claim is fine. Let $M \to N$ a map of complete modules. Seen as complex this is therefore derived complete. But then by Lemma A.20, we get that the kernel and the cokernel of this map is also derived complete. As for extensions, the long exact sequence for Ext-groups concludes.

If M is discrete, as $\Lambda_I M$ is connective concentrated in homological degrees $[0, r]$, the last claim follows from the fact that π_0 is left adjoint when applied to connective objects. ☐

Corollary A.22. *Suppose that* R *is derived* I*-complete. Then any finitely presented module is also derived complete.*

Proof. Closed under finite coproducts=products and cokernels. Finitely presented modules are cokernels of maps $R^m \to R^n$. ☐

Corollary A.23 (Quotients)**.** *Let* $N \subset M$ *be an inclusion of derived* I*-complete modules. Then* M/N *is also* I*-derived complete. If* R *is derived complete and* J *is a derived complete ideal (for example, if it is finitely presented ideal), then* R/J *is also derived complete.*

Lemma A.24 (Derived Nakayama)**.** *Let* $M \in \mathcal{D}(R)_{I-comp}$*. Then* $M = 0$ *if and only if* $M \otimes R/I = 0$*. Therefore if* $M \to N$ *is a map of derived* I*-complete modules,* $M \to N$ *is an isomorphism if and only if it is after* $(-) \otimes^L R/I$

Proof. It suffices to show that $M \otimes \mathrm{Kos}_n(f_1, \ldots, f_r) = 0$. But $\mathrm{Kos}_n(f_1, \ldots, f_r)$ has bounded I-torsion cohomology. Now [Sta24, Tag 0G1T] concludes.

For the last point, consider $\mathrm{fib}(M \to N) \to M \to N$ in $\mathcal{D}(R)_{I-comp}$ and apply the preceding part. ☐

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