PRESENTABLE CATEGORIES AND ADJOINT FUNCTOR THEOREM

Abstract. The goal is to work out presentable and accessible categories and prove the adjoint functor theorem.

Contents

This short note is devoted to introduce some useful definitions and working out the usual adjoint functor theorem. References are [SGA4, Sections I.8, I.9], [AR94, Chapters 1, 2], [KS06, Section 6] and lowering into 1-categorical versions content of [HTT, Sections 5.4, 5.5]. The goal is to prove the classical adjoint functor Theorem 3.1.

1. Ind-objects and Pro-objects

In what follows, we work with the notion κ-directed poset to allow after for the good notion of *accessibility*. The reader may want to take $\kappa = \aleph_0$ if he just wants to care about Ind-objects.

1.1. Filtered categories.

Definition 1.1 (Regular cardinal)**.** An infinite cardinal κ is said to be *regular* if the category $\mathrm{Set}_{<\kappa}$ has all λ-colimits for all cardinals $\lambda < \kappa$.

In what follows κ denotes a regular cardinal. The following notions make sense for cardinals that are not regular but they reduce to the regular case.

Example 1.2. The cardinal $\aleph_0$ is regular. Any successor cardinal $\aleph_{\alpha+1}$ is regular. The limit cardinal $\aleph_\omega$ is the union of $\aleph_n$ for $n \in \omega$ so it is not regular.

We introduce the notion of Ind-objects and Pro-objects. Given a category $\mathcal{C}$ an Ind-object of $\mathcal{C}$ is an object which is the union of objects of $\mathcal{C}$ where each object is treated as "compact" in a certain sense.

Definition 1.3 (κ-filtered)**.** A category I is κ-filtered if it is non empty and every subcategory J with $|\operatorname{Mor}(J)| < \kappa$ has a cocone in I. Explicitly,

(1) For every set C_i of strictly less than κ objects of $\mathcal{C}$, there is an object C and a morphism $f_i \colon C_i \to C$.
(2) For every set of morphisms $(g_i \colon C_1 \to C_2)$ of cardinality strictly less than κ, then there is an object $\mathcal{C}$ and a morphism $f \colon C_2 \to \mathcal{C}$ such that $f \circ g_i$ is independent of i.

Remark. Note that if $\kappa' > \kappa$ then a κ'-filtered category is κ is filtered.

Remark. In the case $\kappa = \aleph_0$, were we say that the category is *filtered* we can reduce to the case of two objects and a pair of morphisms.

Definition 1.4 (κ-directed)**.** A poset I is said to be κ-directed if it is non-empty and every subset of cardinality $< \kappa$ has an upper bound in I.

Remark. Note that a poset is κ-directed set if and only if it is a κ-filtered category when seeing the poset as a category.

Example 1.5. A poset is $\aleph_0$-directed if and only if it is directed. Indeed it means that any finite number of elements has an upper-bound. Therefore, the notion of κ-directed is a stronger condition on a poset comparing to the notion of directed sets, meaning that they are less κ-directed posets than directed posets. Therefore asking that a category has all κ-directed colimits is weaker than asking that it has all directed colimits (so that that there are more categories enjoying this property).

Example 1.6. $\mathbb{R}$ with the natural order is $\aleph_1$-directed.

Example 1.7. Main examples of κ-filtered categories are κ-directed sets. In fact, one can reduce this case as will show Proposition in 1.11. However important natural constructions are only filtered and not directed so we want to introduce both notions.

The following lemma will turn out to be important in the characterizations of $\operatorname{Ind}_\kappa$-objects.

Lemma 1.8. *Let J be a category such that $|\operatorname{Mor}(J)| < \kappa$. Let I be a κ-directed category. Then for any functor $h \colon I \times J \to \mathrm{Set}$ the natural map*

$$\varinjlim_i \varprojlim_j h(i,j) \to \varprojlim_j \varinjlim_i h(i,j)$$

is an isomorphism.

Proof. We write the limit has an equalizer. So we want to show that the map

$$\varinjlim_i \left(\prod_j h(i,j) \rightrightarrows \prod_{j_1 \to j_2} h(i,j)\right) \longrightarrow \prod_j \varinjlim_i h(i,j) \rightrightarrows \prod_{j_1 \to j_2} \varinjlim_i h(i,j)$$

is both surjective and injective.

For injectivity, taking two elements in the source, we can suppose that they live at a common step i using that I is directed say $(x_{i,j})$ and $(y_{i,j})$ with i fixed. Having the same image means that for every j there is a map in I, say $i \to i_j$, witnessing the equalization of $x_{i,j}$ and $y_{i,j}$. Using that I is κ-filtered, we can first find a common target for all these maps and then we may further equalize all those maps implying that the two elements are equal on the source.

For surjectivity, take an element on the target. The cardinality of indices i appearing in it's image under $\rightrightarrows$ maps is $< \kappa$. Therefore we can suppose as I is κ-directed that everything happens at some step i, which shows surjectivity. $\square$

The following concept is motivated by Lemma 1.10 which proof is clear by inspection if universal properties.

Definition 1.9 (Cofinality)**.** A functor $\varphi\colon J \to I$ between two filtered categories is *cofinal* if

(1) For every $i \in I$ there is a $j \in J$ and a morphism $i \to \varphi(j)$.
(2) For every pair of morphism $f, g\colon i \to \varphi(j)$ there exists a morphism $h\colon j \to j'$ such that $\varphi(h)$ equalizes f and g.

Remark. If φ is fully faithful, then condition (1) implies condition (2).

Lemma 1.10. *Let I and J be filtered categories. Let $\varphi\colon J \to I$ a cofinal functor. Let $F\colon I \to \mathcal{C}$ be a functor. Then the natural map*

$$\varinjlim_I F \to \varinjlim_J (F \circ \varphi)$$

is an isomorphism.

The following proposition is attributed to Deligne in [SGA4, Proposition I.8.1.6]. We copy the idea of proof and adapt it to the context of κ-filtered categories.

Proposition 1.11. *Let I be a small κ-filtered category. Then there exist a κ-directed set J and a cofinal functor $\varphi\colon J \to I$.*

Proof. We can suppose that $\mathrm{Ob}(J)$ has no upper bound: one can reduce to this case using the cofinal functor $I \times \kappa \to I$.

Let J be the poset of subcategories of I of cardinality strictly less than κ who have a *unique* (not up to isomorphism) final object. Define $\varphi\colon J \to I$ by sending a subcategory to it's final object.

(1) It is non-empty, as I is non-empty and a single object with only the identity is a subcategory with a unique final object.
(2) It is κ-directed. Indeed, if we have a subset of J of cardinality strictly less than κ, then we can find an object where all of the final objects maps and then equalizes those maps to form a common subcategory with unique final object that contain all considered sub-categories. (We reduced to the case where $\mathrm{Ob}(I)$ has no upper bound).
(3) We now need to show that $\varphi\colon J \to I$ is cofinal. Note that the existence of an upper bound for each object follows from (1).

$\square$

Corollary 1.12. *A category is κ-filtered complete if and only it is κ-directed complete. A functor commute with κ-filtered colimits if and only if it commutes to κ-directed colimits.*

Corollary 1.13. *Let I be a κ-filtered category. There exists a κ-directed set J and a cofinal functor $J \to I$ if and only there is some small κ-filtered category J' and a cofinal functor $J' \to I$ if and only if* $\mathrm{Ob}(I)$ *has a small cofinal subset.*

Definition 1.14. We say that a κ-filtered category I is *cofinally small* it it satisfies one of the equivalent conditions of Corollary 1.13.

1.2. Ind-objects.

Proposition 1.15. *Let $\mathcal{C}$ be a category. Let F be a presheaf on $\mathcal{C}$. The following are equivalent.*

1. *$\mathcal{C}_{/F}$ is κ-filtered and cofinally small.*
2. *There is a cofinally small κ-filtered category I and a functor $C_{-}\colon I \to \mathcal{C}$ such that $F \cong \varinjlim_I h_{C_i}$.*
3. *There is a κ-directed category I and a functor $C_{-}\colon I \to \mathcal{C}$ such that $F \cong \varinjlim_I h_{C_i}$.*

If $\mathcal{C}$ has colimits of size strictly less than κ,

4. *F turns colimits of size strictly less than κ into limits and $\mathcal{C}_{/F}$ is cofinally small.*

But (1)-(2)-(3) always imply (4).

Remark. In the case $\kappa = \aleph_0$, if $\mathcal{C}$ admits finite colimits, the (4)-th condition says that F is left exact.

Proof. The statement (2) and (3) are equivalent due to Proposition 1.11. As $\varinjlim_{C \in \mathcal{C}_{/F}} h_C \cong F$, we see that (1) implies (2). If (*) holds, the full sub-category spanned by the C_i is cofinal in $\mathcal{C}_{/F}$, implying (1).

To prove the equivalence with (4) we begin by noting that if the equivalent conditions (1)-(2)-(3) hold, then F always turns colimits of size strictly less than κ into limits. As representables functors turn colimits into limits, and that colimits of presheaves are computed pointwise, the claim follows from Lemma 1.8 as seen below.

$$\begin{aligned}
F(\varinjlim_j c_j) &= (\varinjlim_i h_{c_i})(\varinjlim_j c_j) \\
&= \varinjlim_i h_{c_i}(\varinjlim_j c_j) \\
&= \varinjlim_i \varprojlim_j \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(h_{c_j}, h_{c_i}) \\
&= \varprojlim_j \varinjlim_i \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(h_{c_j}, h_{c_i}) \\
&= \varprojlim_j \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(h_{c_j}, \varinjlim_i h_{c_i}) \\
&= \varprojlim_j \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(h_{c_j}, F) \\
&= \varprojlim_j F(c_j).
\end{aligned}$$

If $\mathcal{C}$ has colimits of size strictly less than κ and (4) holds, we want to show (1). If we take a set of objects $(C_j)_{j \in J}$ with J of size strictly less than κ we can consider it's coproduct $C = \bigsqcup_j C_j$. As F turns this colimit into a limit we see that if there is a collection of maps $(x_j\colon C_j \to F)$, then there is a map $(C \to F)$ which is an upper bound for the family. If we have less than κ-morphisms that we need to equalize, we can consider the coequlizer in $\mathcal{C}$ which will give what we want using the same argument as above. □

Remark. Representables functors are in $\mathrm{Ind}_\kappa(\mathcal{C})$ for all κ because $\mathcal{C}_{/C}$ has a final object, namely $\mathrm{id}_C\colon C \to C$.

Remark. In practice we do not like taking Ind_κ of a category which do not consists of κ-compact objects. For example Ind(Set) is the category countable systems of sets and where every object in the image Set $\to$ Ind(Set) is therefore treated as compact. For example, in here the system $(\mathbb{N}_{\leq k})_{k\in\mathbb{Z}}$ has a natural map tp $\mathbb{Z}$ which is not an isomorphism. Namely a map to left hand side correspond to a map to $\mathbb{Z}$ which factors through $\mathbb{N}_{\leq k}$ for some k, where a map to the right hand side correspond just to a map to $\mathbb{Z}$.

Definition 1.16 (Ind_κ-objects)**.** Let $\mathcal{C}$ be a category. The category $\mathrm{Ind}_\kappa(\mathcal{C})$ is defined to be the full sub-category of $\widehat{\mathcal{C}}$ of objects satisfying one of the equivalent conditions of Proposition 1.15. We say that a functor $\mathcal{C}^{\mathrm{op}}$ is Ind_κ-representable if it belongs to $\mathrm{Ind}_\kappa(\mathcal{C})$.

1.3. **Morphisms of** Ind**-objects.** Let $F, G \in \mathrm{Ind}_\kappa(\mathcal{C})$. Say $F = \varinjlim_i h_{C_i}$ and $G = \varinjlim_j h_{C_j}$ be realizations of F and G as κ-filtered colimits. Then

$$\mathrm{Hom}_{\mathrm{Ind}_\kappa(\mathcal{C})}(F, G) = \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathcal{C}}(c_i, c_j).$$

In other words a morphism can be described as a collection of maps $f_i\colon c_i \to c_{j_i}$ such that for any $i_1 \to i_2$, there is some $c_{j_{i_1 i_2}}$ and morphisms $j_{i_1} \to j_{i_1 i_2}$ and $j_{i_2} \to j_{i_1 i_2}$ such that the following commutes

$$\begin{array}{ccccc} c_{i_1} & \xrightarrow{f_{i_1}} & c_{j_{i_1}} & \longrightarrow & c_{j_{i_1 i_2}} \\ \downarrow & & & \nearrow & \\ c_{i_2} & \xrightarrow{f_{i_2}} & c_{j_{i_2}} & & \end{array}$$

where the indicated arrows are those from colimit diagrams. The following lemma says that we can simplify the situation.

Lemma 1.17 (Morphisms of Ind-objects)**.** *Let $F = \varinjlim_{i\in I} h_{c_i}$ and $G = \varinjlim_{j\in J} h_{c_j}$ be Ind_κ-objects. Let $\alpha, \beta\colon F \to G$. Then there is a κ-filtered diagram K and cofinal functors $K \to I$ and $K \to J$ with a natural transformation between them such that α and β are induced by functoriality by the K-colimit.*

Proof. Using the morphism (α, β) in $\mathrm{Ind}_\kappa(\mathcal{C}) \times \mathrm{Ind}_\kappa(\mathcal{C}) = \mathrm{Ind}_\kappa(\mathcal{C} \times \mathcal{C})$, it suffices to show the claim for a single morphism. Define K as the category of triples (i, j, φ) of an object of I in J and a morphism $\varphi\colon c_i \to c_j$. The natural transformation is readily given and the claim follows. $\square$

1.4. **Properties of** Ind**-categories.**

Proposition 1.18 (Exactness properties of $\mathrm{Ind}_\kappa(\mathcal{C})$)**.** *Let $\mathcal{C}$ be a category.*

(1) In the category $\mathrm{Ind}_\kappa(\mathcal{C})$, κ-filtered colimits exist and the functor $\mathrm{Ind}_\kappa(\mathcal{C}) \to \widehat{\mathcal{C}}$ commute to these colimits.
(2) Colimits of size strictly less than κ commute to $\mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{C})$.
(3) Both functors in the composition

$$\mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{C}) \to \widehat{\mathcal{C}}$$

commute to all limits.
(4) If $\mathcal{C}$ is complete, then $\mathrm{Ind}_\kappa(\mathcal{C})$ is complete.

(5) If $\mathcal{C}$ is equivalent to a small category in which colimits of size strictly less than κ are representable, then $\mathrm{Ind}_\kappa(\mathcal{C})$ is complete. In this case

$$\mathrm{Ind}_\kappa(\mathcal{C}) = \mathrm{Fun}_{\kappa-\mathrm{cont}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$$

where $\mathrm{Fun}_{\kappa-\mathrm{cont}}$ denotes functors that turns limits of size strictly less than κ into limits.

Proof. For (1), consider a κ-directed system (F_j) in $\mathrm{Ind}_\kappa(\mathcal{C})$. We consider the colimit in $\widehat{\mathcal{C}}$. We need to show that $\mathcal{C}_{\varinjlim_j F_J}$ is κ-filtered and is cofinally small because each of these categories are cofinally small. Note that $\mathrm{Ob}(\mathcal{C}_{/\varinjlim_j F_J}) = \varinjlim_j \mathrm{Ob}(\mathcal{C}_{/F_j})$, so the cofinally small claim follows. If we take a set of objects of $\mathrm{Ob}(\mathcal{C}_{/\varinjlim_j F_J})$ of cardinality stricly less than κ, has J is κ-directed, without loss of generality we can see these objects mapping in a fixed F_j, so the κ-filtered is also clear because $\mathcal{C}_{/F_j}$ is κ-filtered. The idea for the "coequalizer property" is the same.

For (2), this follows from Lemma 1.8.

In statement (3) the second functor is continuous because limits in $\widehat{\mathcal{C}}$ are computed pointwise meaning that to show that $F \to F_i$ form a limit cone, it suffices to show that $\mathrm{Hom}(C, F) \to \mathrm{Hom}(C, F_i)$ is a limit cone in Set for every object $C \in \mathcal{C}$. But as $C \in \mathrm{Ind}_\kappa(\mathcal{C})$ this is true by assumption if $F \to F_i$ is a limit cone $\mathrm{Ind}_\kappa(\mathcal{C})$. Now because the second functor is continuous-fully faithful and the composition is also continuous it follows that the first is also.

For statement (4), we prove that $\mathrm{Ind}_\kappa(\mathcal{C})$ has equalizers and products. First for equalizer, we use Lemma 1.17 to write an equalized diagram as a colimit of equalizer diagram in $\mathcal{C}$. Take the colimit of the equalizer in $\mathcal{C}$. For products, say $(F_j = \varinjlim_{i_j \in I_j} h_{c_{i_j}})$ is a collection of κ-filtered colimits of representables. Define $K = \prod_{j\in J} I_j$. Then K is κ-filtered because this is a component by component check. But then in $\widehat{\mathcal{C}}$ we have

$$\varinjlim_{(i_j)\in K} \prod_{j\in J} h_{c_{i_j}} = \prod_J \varinjlim_{i_j \in I_j} h_{c_{i_j}},$$

because it is a point-wise calculation of sets.

We now prove (5). In this case, we can identify $\mathrm{Ind}_\kappa(\mathcal{C})$ to functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ that turn limit of size strictly less than κ to limits by Proposition 1.15.(4). But as limits commute with limits, a limit of such continuous functors is still continuous in a similar way. □

The following says that Ind_κ is the *κ-filtered cocompletion.*

Proposition 1.19 (Universal property of $\mathrm{Ind}_\kappa(\mathcal{C})$)**.** *Let $\mathcal{C}$ be a small category. Given any functor $F\colon \mathcal{C} \to \mathcal{D}$ to a category which is κ-filtered complete, then there exist a unique extension $F'\colon \mathrm{Ind}_\kappa(\mathcal{C}) \to \mathcal{D}$ up to isomorphism of functors. In other words, there is an equivalence of categories*

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \cong \mathrm{Fun}_\kappa(\mathrm{Ind}_\kappa(\mathcal{C}), \mathcal{D})$$

where Fun_κ denote the full subcategory of functors that commutes with κ-filtered colimits.

Proof. Say $F \in \mathrm{Ind}_\kappa(\mathcal{C})$. Recall that functorially $F \cong \varinjlim_{C\in\mathcal{C}/F} h_C$. Here this colimit κ-filtered because $\mathcal{C}$ is small and by Proposition 1.15. It follows that there is a unique way to extend a functor defined on $\mathcal{C}$ to a functor which preserves κ-filtered colimits. □

Remark. Be careful, if $\mathcal{C}$ is κ-filtered complete, it does not mean that $\mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{C})$ is an equivalence. This is true if and only if every object of $\mathcal{C}$ is κ-compact. But being at the same time composed uniquely of κ-compact objects and being κ-filtered complete is at the same time are often incompatible.

1.5. Pro-objects.

We now quickly define Pro-objects by op-duality.

Definition 1.20 (Pro-objects)**.** Let $\mathcal{C}$ be a category. The category $\mathrm{Pro}_\kappa(\mathcal{C})$ is defined by $\mathrm{Ind}_\kappa(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$. We say that a functor $\mathcal{C} \to \mathrm{Set}$ is Pro_κ-*representable* if it belongs to $\mathrm{Pro}_\kappa(\mathcal{C})$.

Example 1.21. The category Pro(FinSet) is equivalent to compact totally discontinous topological spaces. The category Ind(Pro(FinSet)) is equivalent to locally compact totally discontinous topological spaces.

2. Accessibility and presentability

Definition 2.1 (Accessible functor)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories such that $\mathcal{C}$ has all κ-filtered colimits. Then a functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be κ-*accessible* if F commutes to all κ-filtered colimits. We say that F is *accessible* if the above holds for some κ.

Definition 2.2 (Compact objects)**.** Let $\mathcal{C}$ be a category. An object $C \in \mathcal{C}$ is said to be κ-*compact* if $\mathrm{Hom}(C, -)\colon \mathcal{C} \to \mathrm{Set}$ is κ-accessible. We say that C is *acessible* if the the above holds for some κ.

If h_C commutes to filtered colimits , we say that C is of *finite presentation,* $\aleph_0$-*compact* or simply *compact.*

The following is immediate using Lemma 1.8.

Lemma 2.3. *Let J be a category with $|\mathrm{Mor}(J)| < \kappa$. A colimit over J of κ-compact objects is still κ-compact.*

Remark. In the category of rings or in the category of R-modules for a ring R, the preceding definition agrees with the usual one. If $\kappa = \aleph_1$ then $R[x_i]_{i\in\mathbb{N}}$ is $\aleph_1$-compact. So it extends the notion of "finitely presented" to a larger notion: objects which have at most κ generators and κ relations. This can have some pertinence has one may use some categories which are generated not only but finitely generated objects but only by "bigger" objects. Here is then the definition that goes along the above philosophy.

Definition 2.4 (Accessible category)**.** A category $\mathcal{C}$ is said to be κ-*accessible* if

(1) It has all κ-filtered colimits.
(2) It is generated by κ-filtered colimits by a small full subcategory of κ-compact objects.

We say that $\mathcal{C}$ is *accessible* if the above hold for some κ.

Remark. Equivalently, $\mathcal{C}$ is the κ-filtered cocompletion $\mathrm{Ind}_\kappa(\mathcal{C}_0)$ of a small category $\mathcal{C}_0$. Note that representables objects of $\mathrm{Ind}_\kappa(\mathcal{C}_0)$ are κ-compact.

Remark. Note that it does not follow that a κ-accessible category is κ'-accessible for $\kappa' > \kappa$. Being generated by κ'-colimits is a stronger condition (because there are less κ'-filtered diagrams). This in fact false! See [AR94, Remark 2.13.(8)]. However, one can find arbitrary large cardinals κ' such that an accessible category is κ'-accessible. See [AR94, Lemma 2.14].

Example 2.5. The category Set is $\aleph_0$-accessible. Indeed any set is the colimit of the filtered set of it's finite sets, and finite sets which are in fact exactly the objects of finite presentation in Set. The same holds for the category of R-algebras or R-modules for any ring R. Any topos is accessible.

Here is the main really useful definition.

Definition 2.6. A category $\mathcal{C}$ is said to be *presentable* if it is cocomplete and accessible.

Example 2.7. Any topoi is presentable. Any category of R-algebras or of R-modules for R a (non-commutative) ring.

Recall the following basic lemma.

Lemma 2.8 (Ninja adjoint theorem). *Let $\mathcal{C}$ be a cocomplete category and $\mathcal{A}$ be a small category, and $s\colon \mathcal{A} \to \mathcal{C}$ a functor. Then there is a functor $s_!\colon \widehat{\mathcal{A}} \to \mathcal{C}$,*

$$\begin{tikzcd} \widehat{\mathcal{A}} \arrow[r, "s_!"] & \mathcal{C} \\ \mathcal{A} \arrow[u, "h_\bullet"] \arrow[ur, "s"'] & \end{tikzcd}$$

that extends the functor s, which is left adjoint $s_! \dashv s^*$ to the functor

$$s^*\colon \mathcal{C} \longrightarrow \widehat{\mathcal{A}}$$

$$C \longmapsto \mathrm{Hom}_{\mathcal{C}}(s(-), C)$$

Proof. Let $s_!(F) = \varinjlim_{(x,A)\in\mathcal{A}_{/F}} s(A)$ – this is well defined because $\mathcal{C}$ is a cocomplete category and $\mathcal{A}_{/F}$ is small. We therefore have the following sequence of natural isomorphisms in $F \in \widehat{\mathcal{A}}$ and $C \in \mathcal{C}$,

$$\begin{aligned}
\mathrm{Hom}_{\widehat{\mathcal{A}}}(F, s^*(C)) &\cong \mathrm{Hom}_{\widehat{\mathcal{A}}}(\varinjlim_{(x,A)\in\mathcal{A}_{/F}} h_A, s^*(C)) \\
&\cong \varprojlim_{(x,A)\in\mathcal{A}_{/F}} \mathrm{Hom}_{\widehat{\mathcal{A}}}(h_A, s^*(C)) \\
&\cong \varprojlim_{(x,A)\in\mathcal{A}_{/F}} \mathrm{Hom}_{\mathcal{C}}(s(A), C) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(\varinjlim_{(x,A)\in\mathcal{A}_{/F}} s(A), C) = \mathrm{Hom}_{\mathcal{C}}(s_!(F), C)
\end{aligned}$$

And therefore all our claims are shown. $\square$

Note that if $\mathcal{C}$ is presentable, and $\mathcal{C}_0$ a κ-generating full sub-category of κ-accessible objects, $\mathcal{C}_0 \to \mathrm{Ind}_\kappa(\mathcal{C}_0)$ is a functor like the functor s in the statement of last lemma.

Corollary 2.9. *A category is presentable if and only if it is an accessibly embedded full reflexive sub-category of a free cocompletion of a small category.*

Remark. As being κ-accessibly embedded implies being κ'-accessibly embedded for $\kappa' > \kappa$, the subtlety mentioned above for accessible categories disappears for presentable categories.

Remark. This equivalent formulation can be a motivation for this seemingly convoluted definition: a topos is a category of exactly the same kind, but we ask additionally that the reflector is left exact.[1]

Remark. If $\mathcal{C}$ is presentable, then there is some cardinal κ such that $\mathcal{C} = \mathrm{Ind}_\kappa(\mathcal{C}_0)$ for $\mathcal{C}_0$ a full subcategory of κ-compact objects. As $\mathcal{C}$ is *cocomplete*, we can furthermore describe this category as the full sub-category of $\widehat{\mathcal{C}_0}$ consisting of functors that preserve all κ-small limits existing in $\mathcal{C}$.

3. Adjoint functor theorems

3.1. Adjoint functor theorem for presentable categories.

Theorem 3.1 (Adjoint functor theorem). *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between presentable categories. Then*

(1) F is a left adjoint if and only if it preserves colimits.
(2) F is a right adjoint if and only if it is accessible and preserves limits.

Remark. The theorem is not op-symmetric because hypothesis are not. There is no reason for the opposite of a presentable category to be presentable. Statement (1) and statement (2) are therefore not dual statements. Statement (1) is fairly immediate, but statement (2) is a bit less.

Proof. Let $\mathcal{D}_0, \mathcal{C}_0$ be a full small subcategories of κ-generating κ-compact objects that admit all κ-small colimits such that F is κ-accessible if we assume hypothesis (2).

We prove (1). Note that if F is a left adjoint, it is cocontinuous, so there is only one direction to prove. For a fixed $D \in \mathcal{D}$ we consider $G_D\colon \mathcal{C}_0^{\mathrm{op}} \to \mathrm{Set}$ defined as $G_D(C) = \mathrm{Hom}_\mathcal{D}(F(C), D)$. We want to show that this functor turns limits of size strictly less than κ in $\mathcal{C}_0^{\mathrm{op}}$ into limits. This follows from the cocontinuity of F and from Proposition 1.18 (3).

We prove (2). Suppose that F is accessible and preserves limits. Let $D \in \mathcal{D}_0$. We denote by $G_D\colon \mathcal{C} \to \mathrm{Set}$ the functor defined by $G_D(C) = \mathrm{Hom}_\mathcal{D}(D, F(C))$. Note that as F preserves limits, is κ-accessible and D is κ-compact, G_D is also continuous and preserves κ-filtered colimits; this the property we are going to use below.

We will show that $(\mathcal{C}_{0,G_D/})^{\mathrm{op}}$ is cofinal in $(\mathcal{C}_{G_D/})^{\mathrm{op}}$. Note that both of these categories are κ-filtered filtered because F preserves limits of size stricly less than κ. Indeed, say $(G_D \to C_i)$ is a collection of maps of size stritly less than κ. As

$$\mathrm{Hom}_\mathcal{D}(D, F(\prod_i C_i)) = \prod_i \mathrm{Hom}_\mathcal{D}(D, C_i)$$

we have a factorization $G_D \to \prod_i C_i \to C_i$. The equalizer property is proved in the same way.

As for the cofinality claim, as any object $C \in \mathcal{C}$ is a κ-filtered colimits of κ-compact objects in $\mathcal{C}_0$, say $C = \varinjlim_i C_i^0$ and that G_D preserves κ-filtered colimits, we see that if we have a map $G_D \to C$, using compacity

$$\mathrm{Hom}_\mathcal{D}(D, C) = \mathrm{Hom}_\mathcal{D}(D, \varinjlim_i C_i^0) = \varinjlim_i \mathrm{Hom}_\mathcal{D}(\mathcal{D}, C_i^0).$$

[1]Note that a topos is always presentable. It is cocomplete, and if $\mathcal{C}_0$ is a site and $\kappa > |\mathrm{Mor}(\mathcal{C}_0)|$ then any κ-filtered pointwise colimit of sheaves is again a sheaf. The proof is analogous to the proof that on a quasi-compact topos, a pointwise filtered limit of sheaves is already a sheaf.

In other words there is a factorization $G_D \to C_i^0 \to C$ for some i.

Now, as $\mathcal{C}$ is complete one can take the limit $\varprojlim_{C \in \mathcal{C}_{0,G_D/}} C$ in $\mathcal{C}$, and this gives a initial element of $\mathcal{C}_{G_D/}$, *i.e.* a representing element for G_D. Now one can define $G\colon \mathcal{D}_0 \to \mathcal{C}$ which will extend by continuity to a left adjoint $\mathcal{D} \to \mathcal{C}$.

If F has a left adjoint G, we want to show that F is accessible. Note that for any object $C \in \mathcal{C}$, the value $F(C) \in \mathrm{Ind}_\kappa(\mathcal{D}_0) = \mathrm{Fun}_{\kappa-\mathrm{cont}}(\mathcal{D}_0^{\mathrm{op}}, \mathrm{Set})$ can be described as the functor $\mathcal{D}_0^{\mathrm{op}} \to \mathrm{Set}$ defined by

$$F(C) = \mathrm{Hom}_{\mathcal{C}}(G(-), C).$$

As $\mathcal{D}_0$ is small the essential image of G in $\mathcal{C}$ is small. Therefore as $\mathcal{C}$ is generated by colimits by $\mathcal{C}_0$ there is a regular cardinal λ such that each object of the image of G is λ-compact by Lemma 2.3. Therefore, we see that F is λ-accessible. □

Remark. A functor who preserves limits between accessible categories does not necessarily have a left adjoint. The *accessible* hypothesis is necessary as shows the following example due to H. Bass (*cf.* [SGA4, Remark I.1.8.12.9]). Let I be the *large* collection of isomorphisms classes of simple groups. Consider a large collection $(G_i)_{i \in I}$ where each G_i is a representative of an isomorphism class of simple groups. Let $\mathbb{J}$ be the poset of small subsets of I. Let for $J \in \mathbb{J}$ the group G_J be the coproduct of all groups G_i for $i \in J$. Then $(G_J)_{J \in \mathbb{J}}$ form a projective system of groups with corresponding functor

$$F(G) = \varinjlim_{J \in \mathbb{J}} \mathrm{Hom}_{\mathrm{Grp}}(G_J, G).$$

Note that this functor is set valued, because for each group G there is only a small collection of the simple groups (G_i) who maps non-trivially to G. Even more, for each small subcategory of the category of groups, only a small collection of the simple groups (G_i) will map non-trivially to each group of the small subcategory that we considered. In consequence, the restriction of F to each small subcategory of the category of groups is representable. Therefore F defines a limit preserving functor $\mathrm{Grp} \to \mathrm{Set}$, which are both $\aleph_0$-presentable categories. However, one easily sees that such a functor can not be representable, in particular it can not have a left adjoint. As a consequence, this functor can not be accessible.

3.2. **Pro-adjoint functor theorem.** We now prove a variation of the above theorem. Before, stating it we make the pertinent definition. Recall that $\mathrm{Pro}(\mathcal{C})$ are inverse systems (=cofiltered diagrams) of objects in $\mathcal{C}$ with maps being defines as

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}((C_i), (D_j)) = \varprojlim_j \varinjlim_i \mathrm{Hom}_{\mathcal{C}}(C_i, D_j).$$

Recall that a functor $F\colon \mathcal{C} \to \mathcal{D}$ can be extend uniquely into an inverse limit preserving functor $\mathrm{Pro}(F)\colon \mathrm{Pro}(\mathcal{C}) \to \mathrm{Pro}(\mathcal{D})$. This can be seen for example with Proposition 1.19 and op-duality.

Definition 3.2 (Pro-left adjoint)**.** We say that a functor $F\colon \mathcal{C} \to \mathcal{D}$ has a *pro-left adjoint* if $\mathrm{Pro}(F)$ has a left adjoint.

Remark. This is equivalent to saying that for every $D \in \mathcal{D}$ the functor $\mathcal{C} \to \mathrm{Set}$

$$G_D\colon C \mapsto \mathrm{Hom}_{\mathcal{C}}(D, F(C))$$

is pro-representable. Also, this is equivalent to the existence of a functor $G' \colon \mathcal{D} \to \mathrm{Pro}(\mathcal{C})$ with a natural isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(D, F(C)) \cong \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(G'(D), C).$$

Theorem 3.3 (Pro-adjoint functor theorem)**.** *Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor between presentable categories. Then F has a pro-left adjoint if and only if it is accessible and preserves finite limits.*

Proof. The ideas of the proof are actually contained in the proof of Theorem 3.1.

Say $\mathcal{C}_0$ and $\mathcal{D}_0$ are generating categories of κ-compact objects for $\mathcal{C}$, that we may take containing colimits of size strictly less than κ if needed. Take κ also such that F is κ-accessible and sends κ-compact objects to κ-compact objects.

As any $D \in \mathcal{D}$ is κ-filtered colimit of κ-compact objects D_i^0, say $\varinjlim_i D_i^0 = D$ and that $\mathrm{Pro}(\mathcal{C})$ is cocomplete by Proposition 1.18.(4), it suffices to check that for any $D \in \mathcal{D}_0$, the functor G_D is pro-representable, indeed

$$\mathrm{Hom}_{\mathcal{D}}(D, F(-)) = \mathrm{Hom}_{\mathcal{D}}(\varinjlim_i D_i^0, F(-)) = \varprojlim_i \mathrm{Hom}_{\mathcal{D}}(D_i^0, F(-)) = \varinjlim_i G_{D_i^0}.$$

Showing that $\mathcal{C}_{G_D/}$ is cofiltered uses that F preserves finite limits, as in the proof of Theorem 3.1. Now if $D \in \mathcal{D}_0$, then one can show that $\mathcal{C}_{0,G_D/}$ is final inside this category using the accessibility of F again as in the proof of Theorem 3.1. Therefore this shows that $G_D \in \mathrm{Pro}(\mathcal{C})$, when $D \in \mathcal{D}_0$.

If there is a pro-adjoint, we want to show that F preserves finite limits and is accessible. We use

$$\mathrm{Hom}_{\mathcal{D}}(D, F(C)) \cong \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(G(D), C).$$

Because $\mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ commute with finite limits by Proposition 1.18.(2), we see that F commutes with finite limits. For the accessibility, note that for any object $C \in \mathcal{C}$, the value $F(C) \in \mathrm{Ind}_\kappa(\mathcal{D}_0) = \mathrm{Fun}_{\kappa-\mathrm{cont}}(\mathcal{D}_0^{\mathrm{op}}, \mathrm{Set})$ can be described as the functor $\mathcal{D}_0^{\mathrm{op}} \to \mathrm{Set}$ defined by

$$F(C) = \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(G(-), C).$$

Because $\mathcal{D}_0$ is small, we can assume that there is a λ such that for any $D_0 \in \mathcal{D}_0$ we have $G(D_0) = \varprojlim_i h_{C_i}$ where C_i is λ-compact. Then for any filtered λ-colimit $\varinjlim_j C_j$ we have, for any $D_0 \in \mathcal{D}_0$, using compacity where needed

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(D_0, F(\varinjlim_j C_j)) = F(\varinjlim_j C_j)(D_0) &= \mathrm{Hom}_{\mathrm{Pro}(\mathcal{C})}(G(D_0), \varinjlim_j C_j) \\
&= \varinjlim_i \mathrm{Hom}_{\mathcal{C}}(C_i, \varinjlim_j C_j) \\
&= \varinjlim_i \varinjlim_j \mathrm{Hom}_{\mathcal{C}}(C_i, C_j) \\
&= \varinjlim_j \varinjlim_i \mathrm{Hom}_{\mathcal{C}}(C_i, C_j) \\
&= \varinjlim_j \mathrm{Hom}_{\mathcal{C}}(G(D_0), C_j) \\
&= \varinjlim_j F(C_j)(D_0) \\
&= \varinjlim_j \mathrm{Hom}_{\mathcal{C}}(D_0, F(C_j)) = \mathrm{Hom}_{\mathcal{C}}(D_0, \varinjlim_j F(C_j))
\end{aligned}$$

□

Example 3.4. Consider a geometric morphism of topoi $f\colon \mathcal{E} \to \mathcal{E}'$. Then f^* is accessible being a left adjoint, but it also preserves finite limits by definition. Therefore, there always exist an exceptional pro-direct image

$$f_!\colon \mathcal{E} \to \mathrm{Pro}(\mathcal{E}').$$

Applying this to the unique geometric morphism $f\colon \mathcal{E} \to \mathrm{pt}$ gives the pro-π_0 of a topos. It suffices to explain $f_! 1_{\mathcal{E}}$. We want to see what is the pro-set representing

$$\mathrm{Hom}_{\mathcal{E}}(1_{\mathcal{E}}, f^*S).$$

Given a map $1_{\mathcal{E}} \to f^*S$ pullbacking $f^*s \to f^*S$ for any $s \in S$ gives a disjoint decomposition $\coprod_s U_s = 1_{\mathcal{E}}$ by sub-objects. Therefore we see that the *pro-system of disjoint decomposition by sub-objects of* $1_{\mathcal{E}}$ is the pro-set representing $f_! 1_{\mathcal{E}}$. Therefore, we see that if $\mathcal{E}$ is *locally connected*, implying that every object decomposes a disjoint decomposition of connected subobjects, then this pro-system is constant and f^* has a genuine left-adjoint. We see by Theorem 3.1 that this is the case if and only if f^* preserves all limits.

Example 3.5. Consider a ring A and $f \in A$. Then $j^* = - \otimes A_f\colon \mathrm{Mod}(A) \to \mathrm{Mod}(A_f)$ is accessible being a left adjoint, and preserves finite limits being exact. In consequence we have pro-left adjoint

$$j_!\colon \mathrm{Mod}(A_f) \to \mathrm{Pro}(\mathrm{Mod}(A)).$$

By coccontinuity, we just explain where is sent the generator A_f. Take any $M \in \mathrm{Mod}(A)$. Then

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Mod}(A_f)}(A_f, M_f) &= M_f \\
&= \varinjlim(M \xrightarrow{f} M \xrightarrow{f} \cdots) \\
&= \varinjlim(\mathrm{Hom}_{\mathrm{Mod}(A)}(A, M)) \xrightarrow{f} \mathrm{Hom}_{\mathrm{Mod}(A)}(A, M) \xrightarrow{f} \cdots)
\end{aligned}$$

which implies that the pro-system $(\cdots \xrightarrow{f} A \xrightarrow{f} A)$ is the one representing $j_! A$.

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