

PERFCTOID AND PRISMATIC METHODS

3. δ AND PD-ENVELOPES, ARC TOPOLOGY AND FRIENDS

Exercise 3.1. Let A be a p -torsion free δ -ring over $\mathbb{Z}_{(p)}$. Denote by $\gamma_n(x) = \frac{x^n}{n!}$. Let $k \geq 1$. Show that $\gamma_{pk}(x) \in A$ if and only if $\gamma_k(\gamma_p(x)) \in A$.

Exercise 3.2. Consider $\mathbb{Z}_{(p)}\{x\}[\frac{x^n}{n!}]$. Show that $x_i \mapsto x_i^p + x_{i+1}$ the lift of Frobenius extends uniquely to this ring and gives a Frobenius lift, and therefore a δ -structure.

Exercise 3.3. *Perfectoid rings are reduced.* Let R be a perfectoid ring. Using that both maps

$$(-)^p: R/\varpi R \rightarrow R/pR \quad (-)^p: R[\varpi] \rightarrow R[p]$$

are isomorphisms, show that R is reduced.

Exercise 3.4. Let R be a perfectoid ring and $I = (i_\alpha)_{\alpha \in A}$ where i_α^{1/p^n} exists. Show that any map $R/I \rightarrow S$ where S is perfectoid factors through $R/(\overline{i_\alpha^{1/p^\infty}})$. Moreover show that is perfectoid: first show that quotient $A_{\inf}(R)/(\overline{[i_\alpha^{b1/p^\infty}]})$ is $W\left(\left(R/(\overline{i_\alpha^{1/p^\infty}})\right)^b\right)$. Then take the quotient by d .

Exercise 3.5. Let $A \rightarrow B$ be a ring map.

- (1) If $A \rightarrow B$ is faithfully flat then $A \rightarrow B$ is an arc cover.
- (2) If $A \rightarrow B$ is integral and $\text{Spec}(B) \rightarrow \text{Spec}(A)$ surjective, then $A \rightarrow B$ is an arc cover.
- (3) Let $A \rightarrow B$ be integral and an isomorphism away from I . Let F be an arc-sheaf (of sets for simplicity) then show that

$$\begin{array}{ccc} F(A) & \longrightarrow & F(B) \\ \downarrow & & \downarrow \\ F(A/I) & \longrightarrow & F(B/IB) \end{array}$$

Exercise 3.6. *Mayer-Vietoris for perfection (1).* This exercise is in characteristic p .

- (1) Let $A \rightarrow B$ an injective map of perfect rings and $I \subseteq A$ a perfect ideal such that $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is an isomorphism away from I . Show that square below is cartesian.

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A/IA & \longrightarrow & B/IB \end{array}$$

- (2) Let $f: Y \rightarrow X$ be a proper surjective map. Assume f is an isomorphism outside of $Z \subset X$ and let $E = f^{-1}(Z)$. Show that

$$0 \rightarrow \mathcal{O}_{\text{perf}}(X) \rightarrow \mathcal{O}_{\text{perf}}(Y) \oplus \mathcal{O}_{\text{perf}}(Z) \rightarrow \mathcal{O}_{\text{perf}}(E) \rightarrow 0$$

Hint: use [Tag 03GX](#) with the first point.

Exercise 3.7. *Mayer-Vietoris for perfection (2).* Under the hypothesis as in the preceding exercise, point (2), show that

$$0 \rightarrow R\Gamma_{fppf}(X, \mathcal{O}_{\text{perf}}) \rightarrow R\Gamma_{fppf}(Y, \mathcal{O}_{\text{perf}}) \oplus R\Gamma_{fppf}(Z, \mathcal{O}_{\text{perf}}) \rightarrow R\Gamma_{fppf}(E, \mathcal{O}_{\text{perf}}) \rightarrow 0$$

As a consequence, see [Bhatt & Scholze \(2015\)](#) Theorem 4.1 and its proof if $A \rightarrow B$ is a v-cover of perfect schemes then the associated Čech complex is exact. To get to arc covers one need to see another property, see [Bhatt & Mathew \(2018\)](#) Theorem 4.1.

Exercise 3.8. *aic-v-excision.* Let V be an absolutely integrally closed perfect valuation ring. Let $\mathfrak{p}$ be any prime ideal. Then there is a pullback square.

$$\begin{array}{ccc} V & \longrightarrow & V/\mathfrak{p} \\ \downarrow & & \downarrow \\ V_{\mathfrak{p}} & \longrightarrow & k(\mathfrak{p}) \end{array}$$