

Perfectoidization: Andre's lemma arc topology, almost purity

Recall from first lecture

$$A \text{ } \delta\text{-ring} \quad A\{x\} = A[x_0, x_1, \dots] \quad = \delta(x_0)$$

Free δ -ring on 1 variable.

1) Andre's lemma

Def (Prism envelope) Let $(A, (d))$

prism. Let $a \in A$.

$$\begin{array}{ccc} A\{x\} & \xrightarrow{x \mapsto dx-a} & A\{x\} \\ x \mapsto 0 \downarrow & & \downarrow \\ A & \xrightarrow{\quad} & A\left\{\frac{a}{d}\right\} \end{array} \quad \leftarrow \text{prism envelope of } a$$

(p, d) -complete derived pushout.

Thm ("Hodge-Tate") Say $d=p$

$$A\left\{\frac{\phi(a)}{p}\right\} \cong \mathbb{Z}_p\left\langle \frac{\epsilon^n}{n!} \right\rangle \hat{\otimes}_A^L A$$

Proof Univ case $A = \mathbb{Z}_p\{x\}$ suffices.

Claim: A p -tors free δ -ring $\mathbb{Z}_p$

$$\text{Let } x \in A \quad \gamma_p(x) = \frac{x}{p!} \in A$$

$$\text{Then } \gamma_n(x) \in A \quad \forall n \geq 1$$

Proof Observe $\forall k \geq 1$

$$\gamma_{k \cdot p}(x) \in A \Leftrightarrow \gamma_k(\gamma_p(x)) \in A$$

(Exercises)

$$\text{Now: } \gamma_p(\gamma_p(x)) \in A \quad (*)$$

$$\Leftrightarrow \frac{x^p}{p^{p+1}} = \frac{1}{p} \left(\frac{x^p}{p} \right)^p \in A$$

But:

$$\left(\frac{x^p}{p} \right)^p = \phi\left(\frac{x^p}{p}\right) - p\delta\left(\frac{x^p}{p}\right)$$

$$\text{So } (*) \Leftrightarrow \frac{1}{p} \phi\left(\frac{x^p}{p}\right) \in A$$

But

$$\begin{aligned} & \frac{1}{p^2} (x^p + p\delta(x^p))^p \\ &= p^{p-2} \left(\frac{x^p}{p} + \delta(x^p) \right)^p \in A \end{aligned}$$

$$\text{So } \gamma_p(\gamma_p(x)) \in A$$

Now by induction on n :

$$\forall x \text{ s.t. } \gamma_p(x) \in A \Rightarrow \gamma_n(x) \in A$$

If n divisible by p , induction immediate. $\checkmark$

$$\text{Say } n = kp \quad 0 < k < n$$

$$\gamma_n(x) \in A \Leftrightarrow \gamma_k(\gamma_p(x)) \in A$$

$$\text{But } \gamma_p(\gamma_p(x)) \in A \text{ and induction } \checkmark$$

$\square$ claim

Now the universal case:

$\mathbb{Z}_p\left\{\frac{\phi(x)}{p}\right\}$ is smallest δ -sub-ring of $\mathbb{Z}_p\{x\}[\frac{1}{p}]$ containing $\frac{\phi(x)}{p}$ or $\frac{x^p}{p}$ and $\mathbb{Z}_p\{x\}$.

$$A \text{ s.t. } \frac{\phi(x)}{p} = \frac{x^p}{p} + \delta(x)$$

$$D = \mathbb{Z}_p[x]\left[\frac{x^n}{n!}\right]$$

Have $D \subseteq C$ by claim

Now suffices $\phi: D \rightarrow D$ lift of Frob. (exercises)

$\square$

Corollary Let $A \rightarrow P$ map of

perfect prisms with $\frac{A}{(p, d)}, \frac{P}{(f, p, d)}$

p -torsion free and $f \in P$ with

$$\frac{A}{(p, d)} \rightarrow \frac{P}{(f, p, d)}$$

Faithfully flat.

Then

$$A \rightarrow P\left\{\frac{f}{d}\right\}$$

(p, d) -comp. faithfully flat

Proof Let $D := A\left\{\frac{d}{p}\right\}$

By preceding lemma

$$\frac{D}{p} \cong \frac{A}{(p, d)} \left[\frac{\epsilon_i}{\epsilon_i^p} \right] \quad (PD\text{-envelope modulo } p)$$

$$D \hat{\otimes}_A^L P\left\{\frac{f}{d}\right\} \cong P_b\left\{\frac{f}{p}\right\} \quad (P_b \cong P \hat{\otimes}_A^L D)$$

$$\text{Now } D \rightarrow P_b\left\{\frac{f}{p}\right\}$$

(same lemma) $\downarrow$ modulo p

$$\frac{A}{(p, d)} \left[\frac{\epsilon_i}{\epsilon_i^p} \right] \rightarrow \left(\frac{P}{(p, d, f)} \left[\frac{\epsilon_i}{\epsilon_i^p} \right] \right) \xrightarrow{\gamma_i} \gamma_i^p$$

is faithfully-flat. Now base change

along $D \rightarrow \frac{A}{(p, d)}$ gives the claim. $\square$

Remark Hypothesis imply that every derived step is classical.

Corollary (Andre's lemma) R pfd.

$$m(t) \in R[t] \text{ monic } \exists R' \xleftarrow[p\text{-comp.}]{f\text{f}} R$$

$$\text{s.t. } \exists r' \in R' \quad m(r') = 0, \text{ and}$$

r' admits p -comp. p -power roots.

Proof $(A, (d))$ perfect prism ass. to R

$$m(t) \in A[t] \text{ lift. Now}$$

$$A \rightarrow A\langle \gamma_{p^{\infty}} \rangle \left\{ \frac{m(\gamma)}{d} \right\} \xrightarrow{\frac{A}{(p, d)} \rightarrow \frac{A}{(p, d)} \left[\frac{\gamma_{p^{\infty}}}{\gamma_{p^{\infty}}^p} \right]} \text{ is faithfully flat (free)}$$

Satisfies hyp. of last lemma.

$\rightarrow$ Take perfection and mod d . $\square$

Corollary R pfd $I \subseteq R$ ideal

$$\text{then } \exists R \rightarrow (R/I)_{pfd}$$

$$\downarrow$$

$$R/I$$

initial pfd R -als s.t. $I=0$.

Proof Existence:

$$A \rightarrow A\left\{\frac{I}{d}\right\}_{p\text{ perf}}$$

Surjectivity up to p -comp faithfully flat

b. change

$$I = (i_d)_{d \in A} \quad i_d \text{ admits } p\text{-comp. } p\text{-power roots.}$$

$$\text{In this case } R \xrightarrow{I} (R/I)_{pfd} \text{ (see exercises)}$$

$$\left(\frac{R}{I} \right)_{pfd} = \frac{R}{(i_d^{1/p^{\infty}})} \quad \square$$

Def A semiperfectoid ring is a

derived p -comp ring quotient of

a pfd ring.

2) Arc-topology and perfectoidization

Def A "Valuation ring" is a domain

$$R \text{ s.t. } \forall x \in K = \text{Frac}(R), 0 \\ x \in R \text{ or } x^{-1} \in R.$$

R is of rank ≤ 1 if $\dim(\text{Spec } R) \leq 1$.

Def (p-comp arc topology)

Consider $\mathcal{C} = \{p\text{-comp rings}\}^{\text{op}}$

$R \rightarrow R'$ is a "p-complete arc cover"

If $\forall V$ p-comp rank ≤ 1 val ring

$$\begin{array}{ccc} R & \rightarrow & R' \\ \downarrow & & \downarrow \\ V & \dashrightarrow & V' \end{array} \leftarrow \begin{array}{l} \text{rank } \leq 1 \\ \text{val ring} \end{array}$$

$(R_i \rightarrow R)_{i=1}^n$ is a cover $\Leftrightarrow R \rightarrow \prod R_i$ is

$\leadsto$ Defines p-comp arc topology on $\mathcal{C}$.

Examples • Any p-completion of arc covers

Arc covers (exercises) $A \rightarrow B$

- faithfully flat map

- integral $\text{Spec}(B) \rightarrow \text{Spec}(A)$

Theorem (Bhatt-Matthew-Scholze)

Say $A \rightarrow B$ map of perfect rings which is an arc-cover. Then

$$0 \rightarrow A \rightarrow B \rightarrow B \otimes_A^L B \rightarrow (\dots)$$

exact.

Proof In steps in the exercises $\square$

Lemma • R pfd. $I \subseteq R$ ideal

$$R/I \rightarrow (R/I)_{\text{pfd}} \quad \begin{array}{l} \text{arc-cover} \\ \text{(isomorphism)} \end{array}$$

• A basis for $\mathcal{C}$ is aff pfd.

Proof

$$\begin{array}{ccc} R/I & \rightarrow & (R/I)_{\text{pfd}} \\ \downarrow & \swarrow \text{unique} & \downarrow \\ V & \rightarrow & \bar{V}_{\text{pfd}} \end{array} \quad ; \quad R \rightarrow \tilde{R} \rightarrow (\tilde{R})_{\text{pfd}} \quad \begin{array}{l} \text{semipfd} \\ \text{pfd} \end{array}$$

$\square$

Theorem Let R be a pfd ring.

Then $R \Gamma_{\text{arc}}(e_R, G) = R$.

Proof Suffices if $R \rightarrow R'$ arc-cover

where R' pfd then

$$0 \rightarrow R \rightarrow R' \rightarrow R' \hat{\otimes}_R R' \xrightarrow{(\cdot)}$$

exact.

By tilting equivalence: reduces to the above thm. $\square$

Def Let R be any ring

$$R_{\text{pfd}} := R \Gamma_{\text{arc}}(e_{R^{\text{perf}}}, G)$$

$$= \lim_{\substack{\leftarrow \\ \text{in } \infty\text{-cat. of} \\ \text{derived rings}}} S \quad \begin{array}{l} \leftarrow R \rightarrow S \leftarrow S_{\text{pfd}} \end{array}$$

Remark: If there is an initial pfd ring over R then R_{pfd} is discrete and is this ring.

If R is an algebra over a pfd ring then converse is true.

Corollary Let $R \rightarrow S$ integral

iso away from $I \subseteq R$. Then

$$\begin{array}{ccc} (R)_{\text{pfd}} & \rightarrow & (S)_{\text{pfd}} \\ \downarrow & & \downarrow \\ (R/I)_{\text{pfd}} & \rightarrow & (S/IS)_{\text{pfd}} \end{array} \quad \begin{array}{l} \text{homotopy} \\ \text{pullback} \\ \text{square} \end{array}$$

Proof $R \rightarrow S \times R/I$ arc cover

+ sheaf property

(details in exercises) $\square$

3) Almost purity

Theorem Let R pfd ring
and $R \rightarrow S$ integral. Then
 $(S)_{\text{pfd}}$ discrete.

Lemma Let R any ring.

Then $\left(\frac{R}{\overline{I^2 J}} \right)_{\text{pfd}} \xrightarrow{\sim} \left(\frac{R}{\overline{I J}} \right)_{\text{pfd}}$.

Proof Note $(\overline{I \cap J})^3 \subseteq \overline{I^2 J} \subseteq \overline{I J} \subseteq \overline{I \cap J}$.
 $\Rightarrow$ Same radical.

As perfectoid rings are reduced $\rightarrow \square$

Proof By stability by colimits

and if $\left[\begin{array}{l} S \text{ s.t. } \exists \\ S' \rightarrow S \quad (S')_{\text{pfd}} \text{ discrete} \\ \Rightarrow (S)_{\text{pfd}} \text{ discrete} \end{array} \right]$

So suffices $m(t) \in R[t]$ monic

$\left(\frac{R[t]}{m(t)} \right)_{\text{pfd}}$ discrete.

By André's lemma R AIC

sps $m(t) = \prod_{i=1}^n (t - \alpha_i)$

Let $d = \prod_{i \neq j} (\alpha_i - \alpha_j)$

Then $A \xrightarrow{\text{ev}} \prod_n R$ integral
 t iso away from d .

$\Rightarrow (A)_{\text{pfd}} \rightarrow \left(\prod_n R \right)_{\text{discrete}}$
 $\downarrow \quad \downarrow$
 $\left(\frac{A}{d} \right)_{\text{pfd}} \rightarrow \left(\prod_n \left(\frac{R}{d} \right)_{\text{pfd}} \right)_{\text{discrete}}$

So suffices $d = 0$.

Now $d = (\alpha_1 - \alpha_2) \cdot d'$

and

$A \xrightarrow{(*)} \left(\frac{A}{(\alpha_1 - \alpha_2)} \right)_{\text{pfd}}$ (See next page)
 $\downarrow \quad \downarrow$
 $\left(\frac{A}{d'} \right)_{\text{pfd}} \rightarrow \left(\frac{A}{(\alpha_1 - \alpha_2, d')} \right)_{\text{pfd}}$ (A)

In $(*)$ Have $\frac{\left(\frac{R}{d_1 - d_2} \right)_{\text{pfd}}[t]}{(t - \alpha_1)^2 \cdot (\dots)}$

By lemma: can reduce the degree of $m(t)$.

For d' : Write $d' = (\alpha_i - \alpha_j) \cdot d''$
and run the same argument.

Eventually hit a case: $\frac{R[t]}{(t - \alpha_1)} \cong R \quad \square$

When $R \rightarrow S$ is finite, fin. presented
away from $J = (f_1, \dots, f_n)$:

Theorem (alm purity) $S \rightarrow S_{\text{pfd}}$

iso away from J .

Proof SpS R AIC. SpS $\left(\frac{A}{J} \right)$

$J = (j)$ and $\begin{array}{ccc} R & \rightarrow & S \\ \downarrow & & \downarrow \\ R[\tau_j] & \rightarrow & R[\tau_j]^{xn} \end{array}$ not so immediate

Now $\left(\frac{R^{xn}}{\text{pfd}} \right)$ and $(S)_{\text{pfd}}$ are $(j)_{\text{pfd}}$ -almost iso.

Using: $S \xrightarrow{R^{xn}} S' \xleftarrow{\text{integral closure of both in } R[\tau_j]} S$ $\square$

Corollaire R pfd. Then

$(R[\tau_p])_{\text{set}} \longleftrightarrow (R^b[\tau/\omega^b])_{\text{set}}$.

Proof Can construct

$\xleftarrow{b}$ which is fully-faithful

Last theorem implies ess. surj. $\square$

(A) R any ring I, J ideals

$$\frac{R}{I+J} \rightarrow \frac{R}{J} \quad \text{iso away from } J + \text{integral}$$

$$\begin{array}{ccc} \Rightarrow \left(\frac{R}{I+J} \right)_{\text{pfid}} & \rightarrow & \left(\frac{R}{J} \right)_{\text{pfid}} \\ \downarrow & \searrow & \downarrow \\ \left(\frac{R}{I} \right)_{\text{pfid}} & \rightarrow & \left(\frac{R}{I+J} \right)_{\text{pfid}} \end{array}$$